Example: Uniformly magnetized sphere (hard ferromagnet)

Uniform sphere of radius \( a \)

Effective magnetic charge density

\[
\sigma_m = \hat{n} \cdot M = M_0 \cos \theta
\]

\[
\overline{\sigma}_m = \frac{1}{4\pi} \oint_S \frac{n' \cdot M(x') \, d\alpha'}{|x-x'|} \quad \text{inside}
\]

\[
= M_0 a^2 \frac{1}{4\pi} \oint_S \frac{\cos \theta'}{|x-x'|} \, d\alpha'
\]

\[
\int_{|x-x'|} = \int_{r_{11}}^r \frac{r' P_l (\cos \theta)}{r'^2} \, dr'
\]

\[
\cos \phi = \cos \phi' \cos \theta' + \sin \phi' \sin \theta' \cos (\phi-\phi')
\]

Only

\[
S \, d \Omega' \, \cos \theta' \, P_l (\cos \phi')
\]

Only contributes for \( l=1 \)

\[
S \, \cos \theta' \, \cos \theta' = \frac{2}{3}
\]

\[
\overline{\tau}_m = \frac{M_0 a^2}{3} \frac{r}{r^2} \cos \theta \quad r = \min(c, a)
\]
Inside \( \Phi_m = \frac{m_0}{3} r \cos \theta - \frac{M_0}{3} z \)

\[ H = -\nabla \Phi_m = -\frac{M_0}{3} \hat{z} = -\hat{M} \]

\[ B = \frac{M_0}{3} \left( \hat{H} + \hat{M} \right) = \frac{2}{3} \frac{M_0}{3} \hat{M} \]

Lines of \( B \) are

\[ \text{cont.} \]

Lines of \( H \) end on effective surface charge density

Outside \( r = a \)

\[ \Phi_m = \frac{1}{3} M_0 a^3 \frac{\cos \theta}{r^2} \]

Potential for a magnetic moment

\[ \vec{m} = \frac{4\pi a^3}{3} \hat{M} \]
Magnetized Sphere in External Field: Permanent Magnet.

For uniform sphere:

\[
\vec{B}_\text{in} = \frac{2}{3} \mu_0 \vec{M} \quad \vec{H}_\text{in} = -\frac{1}{3} \vec{M}
\]

Now add external magnetic inductance \(B_0\):

\[
\vec{B}_\text{in} = B_0 + \frac{2}{3} \mu_0 \vec{M} \quad \vec{H}_\text{in} = \frac{1}{\mu_0} B_0 - \frac{1}{3} \vec{M}
\]

IF \(\vec{M}\) is not a permanently magnetized object but is a paramagnetic or diamagnetic substance of permeability \(\mu\)

To find magnitude of \(\vec{M}\) use

\[
\vec{B}_\text{in} = \mu \vec{H}_\text{in}
\]

\[
B_0 + \frac{2}{3} \mu_0 \vec{M} = \frac{\mu}{\mu_0} B_0 - \frac{\mu}{3} \vec{M}
\]

Solve for \(\vec{M}\):

\[
\frac{2}{3} \mu_0 + \frac{\mu}{3} \vec{M} = \left(\frac{\mu - \mu_0}{\mu_0}\right) B_0
\]

\[
\vec{M} = \frac{\mu}{\mu_0 + 2 \mu_0} \left(\frac{\mu - \mu_0}{\mu_0}\right) B_0
\]

Just like polarization \(\vec{P}\) of a dielectric sphere in a uniform electric field:

\[
\vec{P} = 3 \varepsilon_0 \left(\frac{\varepsilon_0 - 1}{\varepsilon_0 + 2}\right) \vec{E}
\]
This does not work because of hysteresis. If we can still eliminate $M_{in}$ from

$$B_{in} = B_0 + \frac{2}{m_0} M_{in} \left( H_{in} + M_{in} - \frac{2}{m_0} B_0 \right)$$

$$H_{in} = \frac{1}{m_0} B_0 - B_0$$

$$B_{in} + 2 \mu_0 H_{in} = 3 B_0$$

hysteresis curve provides another relation

$$F(H_{in}) + 2 \mu_0 H_{in} = 3 B_0$$

Magnetic Shielding.

Consider a spherical shell made of material with permeability $\mu$ in an external field $B_0$.

No currents $J = 0$

$$H = -\nabla \phi_m$$

$$B = \mu H$$

so $\nabla \cdot B = 0$ becomes $\nabla \cdot H = 0$ in different regions.

$$\nabla^2 \phi_m = 0$$

For $r > b$ we have $\phi_m = -H_0 r \cos \theta + \frac{\alpha}{l} \sum_{k=0}^{\infty} \frac{\ell (\cos \theta)^k}{r^{\ell+1}} P_\ell (\cos \theta)$

as $r \to \infty$ this describes uniform $B$ field.

At $r < b$ we have $\phi_m = \frac{\ell}{r}\left( B_0 r^2 + \frac{\alpha}{r^{\ell+1}} \right) P_\ell (\cos \theta)$
\[ r < a \quad \sum m = \sum \frac{1}{r} \rho \left( \cos \theta \right) \]

**Boundary conditions**

\[ (B_2 - \theta_1) \cdot n = 0 \]
\[ \hat{n} \times (\hat{H}_2 - \hat{H}_1) = \nabla \cdot \vec{H} = 0 \]

In radial direction

\[ H_r = -\frac{\partial \vec{A}_m}{\partial \theta} \quad B_r = -\mu \frac{\partial \vec{A}_m}{\partial \theta} \]

\[ \mu_0 \frac{\partial \vec{A}_m}{\partial \theta} = \mu \frac{\partial \vec{A}_m}{\partial \theta} \]

also

\[ \frac{\partial \vec{A}_m}{\partial \theta} = \frac{\partial \vec{A}_m}{\partial \theta} \]

\[ \mu \frac{\partial \vec{A}_m}{\partial \theta} = \mu \frac{\partial \vec{A}_m}{\partial \theta} \]

All coef. except \( k = 1 \) vanish.

\[-H_0 b + \alpha_1 = \beta_1 b + \gamma_1 \]
\[ \alpha_1 - \beta_1 \frac{b^2}{b^2} - \gamma_1 = H_0 b^3 \beta_1 \]

etc.
In limit \( \frac{\mu}{\mu_0} \to \infty \)

\[ \alpha_1 \to b^3 \frac{H_0}{b^3} \]

and \[ -\delta_1 \to \frac{9\mu_0}{2m(1 - \frac{a^3}{b^3})} \frac{H_0}{b^3} \]

So the field inside is much smaller than the field outside going like \( \frac{1}{r} \)

Even for thin shells.

**Faraday's Law of Induction**

Faraday (1831) observed currents in circuits placed in time varying magnetic fields.

The observed transient current is induced in a circuit if

(a) steady current flowing in adjacent circuit is turned on or off,
(b) adjacent circuit with steady current is moved in or out, and
(c) a permanent magnet is thrust into or out of the circuit.

Changing flux induces an electric field around the circuit which is called the electromotive force \( E \). This causes current to flow according to Ohm's law.

Let circuit be bounded by an open surface \( S \) with normal \( \mathbf{n} \). Magnetic flux linking circuit is

\[ \Phi = \int_S \mathbf{B} \cdot \mathbf{n} \, d\mathbf{a} \]
Electromotive force around circuit is

\[ E = \oint E' \cdot dl \]

\( E' \) is electric field at line element \( dl \)

\[ E = -k \frac{dF}{dt} \]

\[ k = 1 \quad \text{V} \cdot \text{C}^{-1} \text{Gaussian units} \]

\[ \oint E' \cdot dl + \oint B \cdot n \cdot da = 0 \]

Use Stokes's thm.

\[ \oint \left( \nabla \times E + \frac{\partial B}{\partial t} \right) \cdot n \cdot da = 0 \]

True for arbitrary surface \( S \)

\[ \nabla \times E + \frac{\partial B}{\partial t} = 0 \]

Time dependent generalization of \( \nabla \times E = 0 \)

Quasi-static Magnetic Fields in Conductors

Eddy currents; Magnetic diffusion

\[ \nabla \times H = J, \quad \nabla \cdot B = 0, \quad \nabla \times E + \frac{\partial B}{\partial t} = 0 \]

\( J = \sigma E \quad \text{Ohm's law} \)
\[ \mathbf{B} = \nabla \times \mathbf{A} \]
\[ \nabla \times (\mathbf{E} + \partial \mathbf{A}/\partial t) = 0 \quad \text{Faraud's law} \]
Therefore, we can define \( \Phi \) so that
\[ \mathbf{E} = -\nabla \Phi - \partial \mathbf{A}/\partial t \]

If free charges act small and time-varying \( \mathbf{B} \) is the sole source of electric field, then \( \Phi = 0 \) and \( \nabla \cdot \mathbf{E} = 0 \) and Coulomb gauge \( \nabla \cdot \mathbf{A} = 0 \)

For media of uniform frequency, independent permeability \( \mu \)
\[ \nabla \times \mathbf{B} = \mu \mathbf{J} \]
\[ \nabla \times \nabla \times \mathbf{A} = \mu \sigma \mathbf{E} = -\mu \sigma \partial \mathbf{A}/\partial t \]
\[ -\nabla^2 \mathbf{A} = -\mu \sigma \partial \mathbf{A}/\partial t \]
\[ \nabla^2 \Phi = \mu \sigma \partial \mathbf{A}/\partial t \]

Take the first time derivative
\[ \nabla^2 \mathbf{E} = \mu \sigma \partial \mathbf{E}/\partial t \]

Can estimate time \( \tau \) for fields to decay
\[ \nabla^2 \mathbf{E} = \mathbf{E} \quad \partial \mathbf{E}/\partial t = \mathbf{E}/c \]
\[ L = \text{length scale} \quad \frac{L^2}{c} \]

\[ \delta \mathbf{E}/\partial t = \mathbf{E}/c \]