Magnetostatics   Lec 17

midterm
ave 81
high 105

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$$dB = k\, I\, d\ell \wedge \vec{x}/x^3 \quad \text{Biot + Savart Law}$$

$$B(x) = \frac{\mu_0}{4\pi} \int J(x') \wedge \frac{\overrightarrow{x - x'}}{|x - x'|^3}\, d^3x'$$

$$= -\frac{\mu_0}{4\pi} \int J \wedge \nabla_x \left(\frac{1}{\overrightarrow{x - x'}} \right) d^3x'$$

$$B = \frac{\mu_0}{4\pi} \vec{\nabla}_x \wedge \int \frac{\vec{J}(x')}{|\overrightarrow{x - x'}|}\, d^3x'$$

$$\nabla \wedge B = \frac{\mu_0}{4\pi} \nabla_x \wedge \nabla_x \wedge \int \frac{J}{|\overrightarrow{x - x'}|}\, d^3x'$$

$$= \frac{\mu_0}{4\pi} \nabla_x \left(\nabla_x \cdot \int \frac{J(x')}{|x - x'|} \right) d^3x'$$

$$- \frac{\mu_0}{4\pi} \nabla_x^2 \int \frac{J(x')}{|x - x'|}\, d^3x'$$

$$= \frac{\mu_0}{4\pi} \nabla_x \int \vec{J}(x') \cdot \vec{\nabla}_x \frac{1}{|\overrightarrow{x - x'}|}\, d^3x'$$

$$- \frac{\mu_0}{4\pi} \int J(x')\, \overbrace{\nabla_x^2 \frac{1}{|x - x'|}}^{-4\pi\,\delta(x - x')}\, d^3x'$$

$$= -\frac{\mu_0}{4\pi} \nabla_x \left[\int J(x') \cdot \nabla_{x'} \left(\frac{1}{|\overrightarrow{x - x'}|} \right) d^3x' \right]$$

$$+ \mu_0 J(x)$$

$$= \frac{\mu_0}{4\pi} \nabla_x \int \nabla_{x'} \cdot J(x') \frac{1}{|x - x'|}\, d^3x' + \mu_0 J$$

$\nabla_{x'} \cdot J(x') = 0$ for static situation

$\propto d\rho/dt$

$$\boxed{\nabla \wedge B = \mu_0 J} \quad (*)$$

analog of $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$

Integral of ⊛ is Ampere's law

$$\int_S (\nabla \wedge B) \cdot n\, da = \mu_0 \int_S \vec{J} \cdot n\, da$$

If contour C bounds surface S then

Stokes's Thm.

$$\boxed{\oint_C B \cdot d\ell = \mu_0 \int_S J \cdot n\, da = \mu_0 I}$$

I = current flowing through surface

Vector potential

How to solve

$$\nabla \wedge B = \mu_0 J$$

$$\nabla \cdot B = 0$$

exploit $\nabla \cdot B = 0$ to define

$$B(x) = \vec{\nabla} \wedge \vec{A}(x)$$

$\vec{A}(x)$ = vector potential

$$A = \frac{\mu_0}{4\pi} \int \frac{J(x')}{|x - x'|} d^3x' + \nabla \Psi(x)$$

Can add arbitrary gradient of ~~sc~~ any scalar function $\Psi(\vec{x})$

$$A \to A + \nabla \Psi(x)$$

gauge transformation, does not change B

$$\nabla \wedge (\nabla \Psi(x)) \equiv 0$$

$$\nabla \wedge B = \mu_0 J = \nabla \wedge (\nabla \wedge A) = \nabla(\vec{\nabla} \cdot \vec{A}) - \nabla^2 A$$

Can choose $\vec{\nabla} \cdot \vec{A} = 0$ gauge choice

then

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

Choice $\nabla \cdot A = 0$ corresponds to Ψ = constant.

$$A = \frac{\mu_0}{4\pi} \int \frac{J(x')}{|x - x'|} d^3x'$$

$$\nabla \cdot A = \frac{\mu_0}{4\pi} \int \vec{J}(x') \cdot (-\vec{\nabla}_{x'}) \frac{1}{|x - x'|} d^3x'$$

$$= \frac{\mu_0}{4\pi} \int \frac{\nabla \cdot J(x')}{|x - x'|} d^3x' = 0$$

Vector Potential from a Circular Current loop.

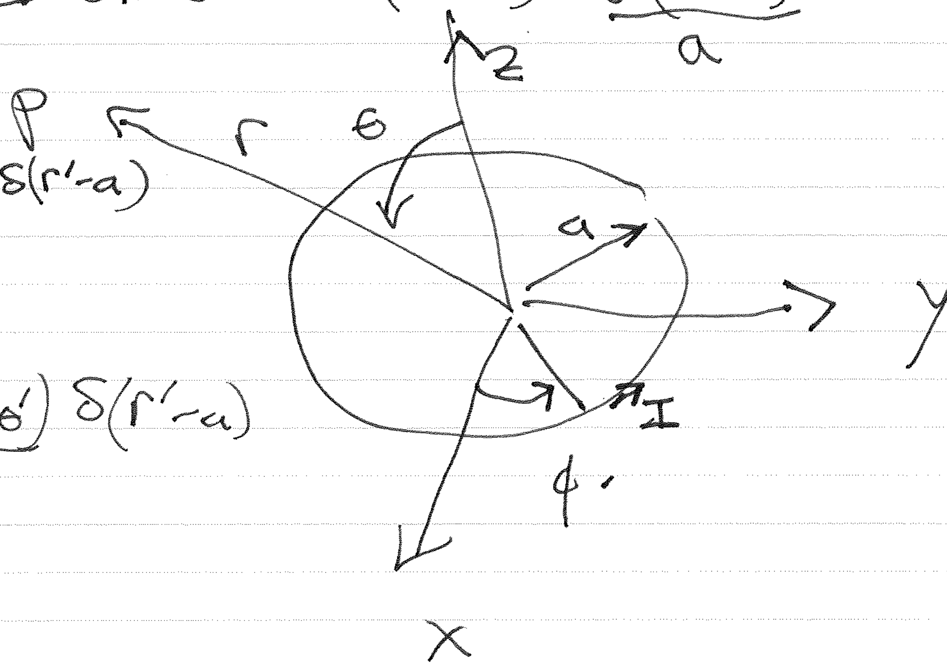
$$J_\phi = I \sin\theta' \, \delta(\cos\theta') \, \frac{\delta(r'-a)}{a}$$

Note

$$\rho = \frac{Q}{2\pi a^2} \delta(\cos\theta') \, \delta(r'-a)$$

$$J = \left(\frac{2\pi a}{T}\right) \rho$$

$$= \left(\frac{Q}{T}\right) \frac{\delta(\cos\theta')}{a} \delta(r'-a)$$



$$\vec{J}_\phi = -J_\phi \sin\phi' \, \hat{x} + J_\phi \cos\phi' \, \hat{y}$$

Choose P to lie in x-z plane ($\phi = 0$)

$$\vec{A} = \frac{\mu_0}{4\pi} \int I \sin\theta' \, \delta(\cos\theta') \frac{\delta(r'-a)}{a} \frac{r'^2 \, dr' \, d\Omega'}{|\vec{x} - \vec{x}'|} \left[\cos\phi' \, \hat{y}' - \sin\phi' \, \hat{x}'\right]$$

$$A_y = A_\phi = \frac{\mu_0}{4\pi} I a \int_0^{2\pi} \frac{\cos\phi' \, d\phi'}{(a^2 + r^2 - 2ar \sin\theta \cos\phi')^{1/2}}$$

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi} \frac{4 I a}{(a^2 + r^2 + 2ar\sin\theta)^{1/2}} \left[\frac{(2-k^2) K(k) - 2E(k)}{k^2}\right]$$

complete elliptic integrals K, E of k

$$k^2 = \frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}$$

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\phi\right)$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left(r A_\phi\right)$$

$$B_\phi = 0$$

For $a \gg r$, $a \ll r$ or $\theta \ll 1$
expand integral in powers of

$$x = \frac{a^2 r^2 \sin^2\theta}{(a^2+r^2)^2}$$

$$A_\phi = \frac{\mu_0 I a^2 r\sin\theta}{4(a^2+r^2)^{3/2}}\left[1 + \frac{15a^2r^2\sin^2\theta}{8(a^2+r^2)^2} + \dots\right]$$

for $r \gg a$

$$A_\phi \simeq \frac{\mu_0 I a^2 \sin\theta}{4 r^2}$$

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{\mu_0 I a^2 \sin^2\theta}{4r^2}\right]$$

$$= \frac{\mu_0 I a^2}{2 r^3}\cos\theta = \frac{\mu_0}{4\pi}\left(I\pi a^2\right)\frac{2\cos\theta}{r^3}$$

$$B_\theta = \frac{\mu_0 I a^2 \sin\theta}{4 r^3} = \frac{\mu_0}{4\pi}\left(I\pi a^2\right)\frac{\sin\theta}{r^3}$$

Define magnetic dipole moment $m = I\pi a^2$

Note Electric field for an electric dipole

$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \qquad E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

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$$E_\phi = 0 \qquad (4.12)$$

$$\vec{p} = \int d^3x' \, \vec{x}' \, \rho(x')$$

$$B_r = \frac{\mu_0}{4\pi} \frac{2m\cos\theta}{r^3}, \quad B_\theta = \frac{\mu_0}{4\pi} \frac{m\sin\theta}{r^3}$$

We can expand in multipoles

$$A_y = A_\phi = \frac{\mu_0}{4\pi} \int I \sin\theta' \, \delta(\cos\theta') \frac{\delta(r'-a)}{a} r'^2 dr'.$$

$$d\cos\theta' \, d\phi' \cos\phi' \sum_{lm} \left(\frac{4\pi}{2l+1}\right) \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta'\phi') Y_{lm}(\theta,0)$$

$$\cos\phi' = \mathrm{Re} \; e^{i\phi'}$$

$$\cancel{Y_{lm}(\theta'\phi') \propto e^{im\phi'}} \qquad Y^*_{lm}(\theta',\phi') = Y_{lm}(\theta',0)\, e^{-im\phi'}$$

$$\int d\phi' \, e^{i\phi'} \, Y^*_{lm}(\theta'\phi') = \delta_{m1} \, 2\pi \, Y_{l1}(\theta',0)$$

$$A_\phi = 2\pi \mu_0 I a \sum_l \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{l1}\left(\frac{\pi}{2},0\right) Y_{l1}(\theta,0)$$

$$r_< = \min(r,a)$$
$$r_> = \max(r,a)$$

$$Y_{l1}\left(\frac{\pi}{2},0\right) = \sqrt{\frac{2l+1}{4\pi l(l+1)}} \; P_l^1(0) \; \cancel{= \left\{ 0 \;\; l \text{ even} \;\; \sqrt{\frac{2l+1}{4\pi l(l+1)}} \right.}$$