No class 10/14/10
Midterm 10/28/10

10/12/10

Lec. 13 Electrostatics of Ponderable Media

If a medium contains many bound charges it is useful to form macroscopic averages.

Average over macroscopicly small region that is microscopicly large (contains many molecules).

The dominant molecular multipole with applied fields is the dipole.

Electric polarization $\vec{P}(x) = \sum_i N_i \langle p_i \rangle$
(dipole moment per unit volume)

$\langle p_i \rangle$ is dipole moment of the ith type of molecule in medium averaged over a small volume centered at $\vec{x}$ and N_i is average number per unit volume of ith type of molecule.

Consider

$$\rho(\vec{x}) = \sum_i N_i \langle e_i \rangle + \rho_{excess}(\vec{x})$$

$\langle e_i \rangle$ = average charge on molecule usually 0.

ρ_{excess} = excess of free charge

Consider contribution of ρ and P from a small volume ΔV to Φ

$$\Delta\Phi(x,x') = \frac{1}{4\pi\epsilon_0}\left[\frac{\rho(x')}{|x-x'|}\Delta V + \frac{P(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\Delta V\right]$$

Used Φ for a point dipole (4.10)

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3}$$

Note ΔV is large enough to contain many molecules but small $\Rightarrow d^3x'$ on macroscopic scale

$$\Phi(x) = \int d^3x\, \Delta\Phi(x,x')$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3x' \left\{\frac{\rho(x')}{|\vec{x}-x'|} + P(\vec{x}')\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right\}$$

$$\nabla'\frac{1}{|x-x'|} = +\frac{1}{(\vec{x}-\vec{x}')^3}(\vec{x}-\vec{x}')$$

Integrate by parts

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{[\rho(x') - \nabla'\cdot P(x')]}{|\vec{x}-\vec{x}'|}$$

customary expression for Φ with effective charge density $\rho - \nabla \cdot P$

$$E = -\nabla \Phi$$

$$\nabla^2 \frac{1}{|x-x'|} = -4\pi\delta(\vec{x}-\vec{x}')$$

$$\nabla \cdot E = -\nabla^2 \Phi = \frac{1}{\epsilon_0}(\rho(x) - \nabla \cdot P)$$

Define electric displacement D

$$D \equiv \epsilon_0 E + P$$

$$\boxed{\nabla \cdot D = \rho}$$

Note still have

$$\boxed{\nabla \wedge E = 0}$$

This is unchanged by polarization

Need a constitutive relation connecting D and E. Note the magnitude of P for a given E depends on how polarizable the molecules are.

Assume response of system is linear

$$\vec{P} \propto \vec{E}$$

and assume medium is isotropic

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

χ_e = electric susceptibility of medium. The displacement D is proportional to E

$$\vec{D} = \epsilon \vec{E}$$

$$\epsilon = \epsilon_0 (1 + \chi_e)$$

$\epsilon/\epsilon_0 = 1 + \chi_e$ is dielectric constant or relative electric permitivity ~~is dielectric constant~~

If dielectric is not only isotropic but uniform - than ϵ is indep. of position

$$\nabla \cdot E = \rho/\epsilon$$

All problems in that medium are reduced to those of proceeding chapters with given charges reduced by a factor ϵ/ϵ_0

Boundary conditions

normal component of D cont.

tangential component of E cont.

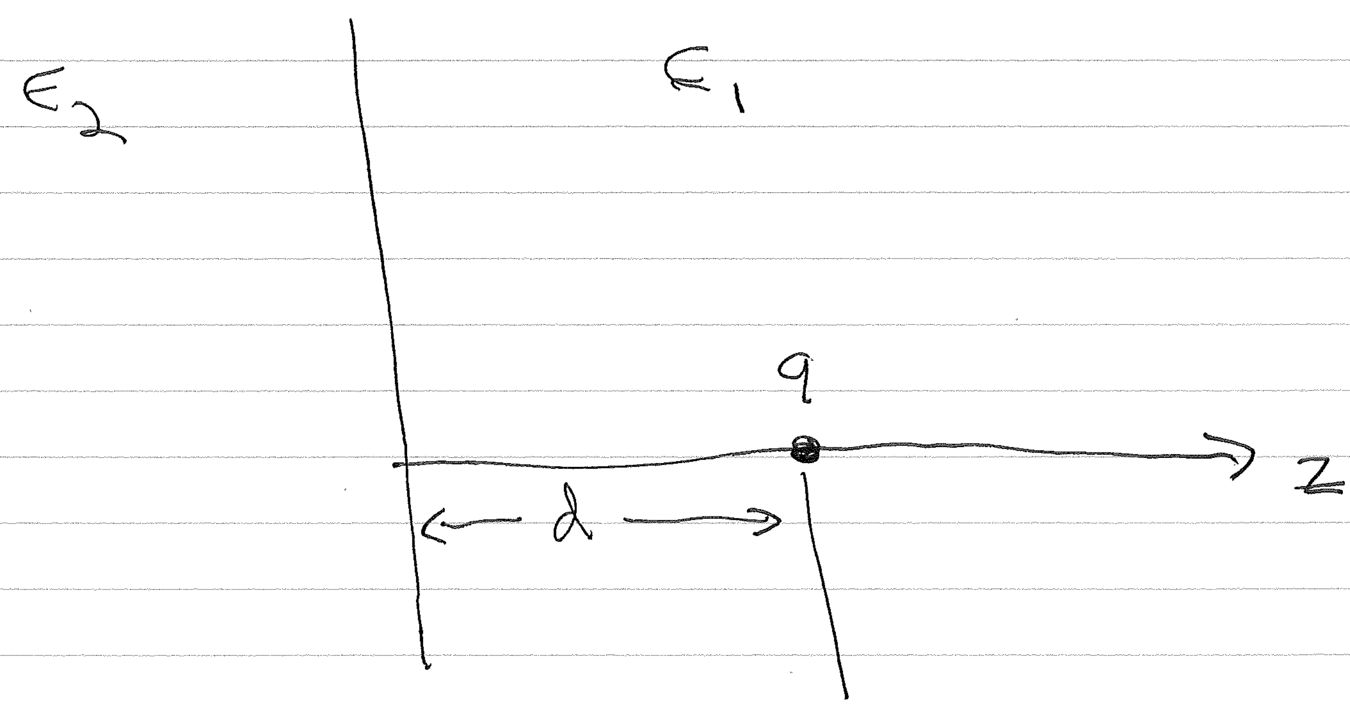
let n_{21} be unit normal to surface directed from region 1 to 2

$$\left(D_2 - D_1\right) \cdot \hat{n}_{21} = \sigma$$

σ = macoscopic ~~surface~~ charge density on surface not including polarization charge

$$\left(\tilde{E}_2 - \tilde{E}_1\right) \wedge \hat{n}_{21} = 0$$

Boundary-Value Problem with Dielectrics



No charges for $z<0$

$$\epsilon_1 \nabla \cdot E = \rho \qquad z > 0$$

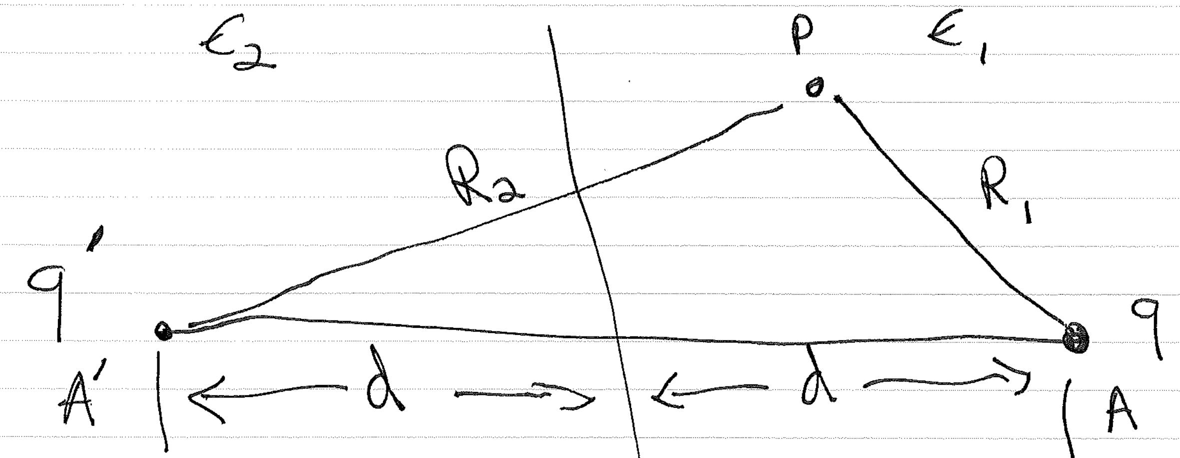
$$\epsilon_2 \nabla \cdot E = 0 \qquad z < 0$$

$$\nabla \wedge E = 0 \qquad \text{everywhere}$$

$$\lim_{z \to 0^+} \begin{Bmatrix} \epsilon_1 E_z \\ E_x \\ E_y \end{Bmatrix} = \lim_{z \to 0^-} \begin{Bmatrix} \epsilon_2 E_{\cancel{x}z} \\ E_x \\ E_y \end{Bmatrix}$$

Since $\nabla \wedge E = 0$ we can derive E from a potential.

Try metod of images



$$\Phi = \frac{1}{4\pi\epsilon_1} \left(\frac{q}{R_1} + \frac{q'}{R_2} \right) \qquad z > 0$$

For $z < 0$ we note that

$$\nabla^2 \Phi = 0 \qquad \text{for all } z < 0$$

so any image charges must be located at $z>0$. Symmetry suggests image charge located at A. Guess

$$\Phi = \frac{1}{4\pi\epsilon_2}\frac{q''}{R_1} \qquad z<0$$

$$R_1 = \sqrt{\rho^2+(d-z)^2} \qquad R_2 = \sqrt{\rho^2+(d+z)^2}$$

cylindrical coordinates along z

$$\frac{\partial}{\partial z}\left(\frac{1}{R_1}\right)_{z=0} = -\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)_{z=0} = \frac{d}{\left[\sqrt{\rho^2+d^2}\right]^{3/2}}$$

$$\epsilon_1 E_z(0^+) = \frac{-1}{4\pi}\left\{\frac{q\,d}{\left[\sqrt{\rho^2+d^2}\right]^{3/2}} - \frac{q'\,d}{(\rho^2+d^2)^{3/2}}\right\}$$

$$\epsilon_2 E_z(0^-) = -\frac{1}{4\pi}\frac{q''\,d}{(\rho^2+d^2)^{3/2}}$$

$$\Rightarrow\ q - q' = q'' \qquad \text{so that } \epsilon_1 E_z(0^+) = \epsilon_2 E_z(0^-)$$

$$\frac{\partial}{\partial \rho}\left(\frac{1}{R_1}\right)_{z=0} = \frac{-\rho}{(\rho^2+d^2)^{3/2}} = \frac{\partial}{\partial \rho}\left(\frac{1}{R_2}\right)_{z=0}$$

$$\frac{1}{4\pi\epsilon_1}(q+q')\left\{\frac{-\rho}{(\rho^2+d^2)^{3/2}}\right\} = \frac{1}{4\pi\epsilon_2}q''\left(\frac{-\rho}{(\rho^2+d^2)^{3/2}}\right)$$

$$\frac{1}{\epsilon_1}(q+q') = \frac{1}{\epsilon_2} q''$$

$$\frac{1}{\epsilon_1}(q+q') = \frac{1}{\epsilon_2}(q-q')$$

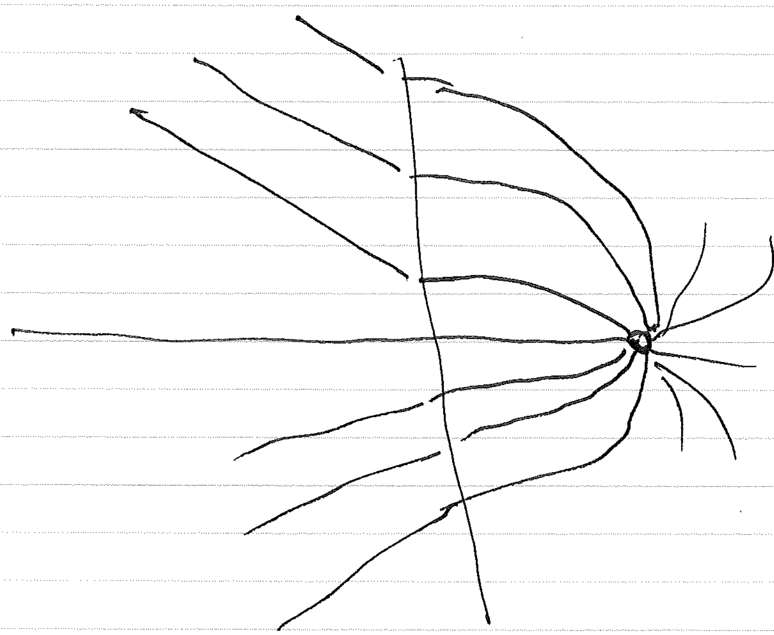
$$\frac{\epsilon_2}{\epsilon_1}(q+q') = q-q'$$

$$\left(1+\frac{\epsilon_2}{\epsilon_1}\right) q' = q\left(1-\frac{\epsilon_2}{\epsilon_1}\right)$$

$$\boxed{q' = q\left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)}$$

$$q'' = \frac{\epsilon_2}{\epsilon_1}\left[1+\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right] q$$

$$\boxed{q'' = \left(\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\right) q}$$



$\epsilon_2 > \epsilon_1$

end 10/12/10

$$D = \epsilon_1 E = \epsilon_0 E + P$$

$$P_1 = (\epsilon_1 - \epsilon_0) E_1$$

There is a surface charge density

$$\sigma_{pol} = -(P_2 - P_1) \cdot n_{21}$$

$$P_2 = (\epsilon_2 - \epsilon_0)\left\{ -\frac{\partial \Phi(0^-)}{\partial z} \right\}$$

$$P_1 = (\epsilon_1 - \epsilon_0)\left\{ -\frac{\partial \Phi(0^+)}{\partial z} \right\}$$

$$P_1 = \frac{(\epsilon_1 - \epsilon_0)}{4\pi\epsilon_1}\left\{ \frac{-qd}{(\rho^2+d^2)^{3/2}} + \frac{q'd}{(\rho^2+d^2)^{3/2}} \right\}$$

$$P_1 = \frac{-q}{4\pi\epsilon_1}(\epsilon_1 - \epsilon_0)\left[1 - \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \right] \frac{d}{(\rho^2+d^2)^{3/2}}$$

$$P_1 = \frac{-q}{2\pi}\left(\frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + \epsilon_2}\right)\left(\frac{\epsilon_2}{\epsilon_1}\right)\frac{d}{(\rho^2+d^2)^{3/2}}$$

$$P_2 = -\frac{q''}{4\pi\epsilon_2}\frac{d}{(\rho^2+d^2)^{3/2}}(\epsilon_2 - \epsilon_0)$$

$$= -\frac{q(\epsilon_2 - \epsilon_0)}{2\pi(\epsilon_1 + \epsilon_2)}\frac{d}{(\rho^2+d^2)^{3/2}}$$

$$(P_1 - P_2) \cdot n_{21} = \frac{-q}{2\pi}\frac{d}{(\rho^2+d^2)^{3/2}}\left[\frac{\epsilon_2}{\epsilon_1}\frac{\epsilon_1 - \epsilon_0}{(\epsilon_1 + \epsilon_2)} - \left(\frac{\epsilon_2 - \epsilon_0}{\epsilon_1 + \epsilon_2}\right) \right]$$

$$= \frac{-q}{2\pi}\frac{d}{(\rho^2+d^2)^{3/2}}\left[\epsilon_0\left(1 - \frac{\epsilon_2}{\epsilon_1}\right)\frac{1}{\epsilon_1 + \epsilon_2} \right]$$