9/30/10

Lec. 10 Green Functions Finish Read

Green's func. in cylindrical coordinates chap 4

$$G(\vec{x},\vec{x}') = \frac{1}{2\pi^2} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\, e^{im(\phi-\phi')} \cos[k(z-z')]\, g_m(k\rho,\rho')$$

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{dg_m}{d\rho} - \left(k^2 + \frac{m^2}{\rho^2}\right) g_m = -\frac{4\pi}{\rho}\delta(\rho-\rho')$$

For $\rho \neq \rho'$, g_m satisfies modified Bessel eq. $\rightarrow$ solutions

$$I_\nu(x) \equiv i^{-\nu} J_\nu(ix) \rightarrow e^{x} \quad x \text{ large}$$
$$K_\nu(x) \equiv \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \rightarrow e^{-x} \quad x \text{ large}$$

$$g_m = \begin{cases} A(\rho') I_\nu(k\rho) + B(\rho') K_\nu(k\rho) & \rho > \rho' \\ A'(\rho') I_\nu(k\rho) + B'(\rho') K_\nu(k\rho) & \rho < \rho' \end{cases}$$

Must be sym. in ρ, ρ' $\quad \nu = m$

$$g_m = \psi_1(k\rho_<)\psi_2(k\rho_>)$$

If b.c. $=0$ at $\rho = 0, \infty$ than

$$g_m = A\, I_m(k\rho_<) K_m(k\rho_>)$$

Choose A

so discont. in derivative satisfies $\delta(\rho-\rho')$

$$\left.\frac{d g_m}{d\rho}\right|_{+} - \left.\frac{d g_m}{d\rho}\right|_{-} = -\frac{4\pi}{\rho'}$$

$+ \Rightarrow \rho = \rho' + \epsilon$

$\psi_{1,2}(x)$ satisfy

$$\frac{d}{dx}\left(x\frac{d}{dx}\right)\psi_i - \left(x + \frac{m^2}{x}\right)\psi_i = 0$$

Wronskian of $\psi_1\ \psi_2$

$$W[\psi_1, \psi_2] = \psi_1 \frac{d\psi_2}{dx} - \psi_2 \frac{d\psi_1}{dx}$$

$$\propto \frac{1}{p(x)} \quad \text{where}$$

$$\frac{d}{dx}\, p(x) \frac{d}{dx}\, y + f(x)\, y = 0$$

so $p(x) = x \qquad W \propto \frac{1}{x}$

let $\psi_1 = I_m(x)$
$\psi_2 = K_m(x)$

W is indep of x can evaluate it anywhere. Conside small x

$$K_m(x) \to \begin{cases} -\left[\ln \frac{x}{2} + .5772\ldots\right] & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^{\nu} & \nu \neq 0 \end{cases}$$

$$J_\nu \to \frac{(x/2)^\nu}{\Gamma(\nu+1)}$$

So

$$W\left[I_m(x), K_m(x)\right] = -\frac{1}{x}$$

So $g_m = 4\pi\, I_m(k\rho_<)\, K_m(k\rho_>)$

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}\cos\left[k(z-z')\right] I_m(k\rho_<)\, K_m(k\rho_>)$$

$$= \frac{4}{\pi}\int_0^\infty dk\, \cos k(z-z') \left\{\frac{1}{2} I_0(k\rho_<) K_0(k\rho_>) + \sum_{m=1}^{\infty} \cos[m(\phi-\phi')]\, I_m(k\rho_<)\, K_m(k\rho_>)\right\}$$

$$I_m(x\to 0) = \delta_{m0}$$

let $x' = 0$

$$\frac{1}{x} = \frac{1}{\sqrt{\rho^2+z^2}} = \frac{2}{\pi}\int_0^\infty dk \cos[k(z)] K_0(k\rho)$$

~~Now let $x \to |\vec{x}-\vec{x}'|$~~

$\Rightarrow$ let $\rho^2 \to R^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi')$

$$\frac{1}{|\vec{x}-\vec{x}'|}\Big|_{z'=0} = \frac{1}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2)^{1/2}}$$

$$= \frac{2}{\pi}\int_0^\infty dk \cos(k z) K_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right)$$

$$= \frac{4}{\pi}\int_0^\infty dk \cos(k(z))$$

$$\left\{\frac{1}{2} I_0 K_0 + \sum_{m=1}^{\infty}\cos(m\phi-\phi') I_m K_m\right\}$$

has to hold for all z $\Rightarrow$

$$K_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right)$$

$$= I_0(k\rho_<) K_0(k\rho_>)$$

$$+ 2\sum_{m=1}^{\infty}\cos[m(\phi-\phi')] I_m(k\rho_<) K_m(k\rho_>)$$

can take limit $k \to 0$ and derive
G for two dim. polar coord.

$$\boxed{\ln \frac{1}{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')} = 2\ln\frac{1}{\rho_>} + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos m(\phi-\phi')}$$

~~In general can write G~~

$$\cancel{G \quad \sum_{l}\sum_{m} f(x)}$$

Note

$$G \propto \sum_{lm} Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta\phi)\, r_<^{l} \frac{1}{r_>^{l+1}}$$

or spherical

$$G \propto \sum_{lm} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi}{a}x'\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi}{b}y'\right)$$

$$\sinh\left(K_{lm} z_<\right) \sinh\left[K_{lm}(c-z_>)\right]$$

for rect. box

$$K^2_{lm} = \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)$$

Eigenfunction expansions for Greenfunctions

$$\nabla^2 \psi(x) + [f(x) + \lambda]\psi(x) = 0$$

$\psi(x) =$ eigenfunction $\quad \lambda =$ eigenvalue

Example schrodinger eq.

In general can only satisfy b.c. for certain values of $\lambda = \lambda_n$

$$\nabla^2 \psi_n + (f(x) + \lambda_n)\psi_n = 0$$

Can show $\int_V \psi_m^* \psi_n \, d^3x = \delta_{mn}$

Now consider Green function

$$\nabla_x^2 G(\vec{x}, \vec{x}\,') + (f(x) + \lambda) G = -4\pi\,\delta(x - x')$$

Look for G with same b.c. and expand

$$G = \sum_n a_n(\vec{x}\,')\,\psi_n(x)$$

$$\cancel{a_m(x')}\ \nabla_x^2 G = \sum_m a_m(x')\,\nabla^2\psi_m$$

$$= -\sum a_m\,(f(\vec{x}) + \lambda_n)\,\psi_n$$

$$\sum_m (\lambda - \lambda_m)\, a_m(\vec{x}\,')\,\psi_m(\vec{x}) \; n = -4\pi\,\delta(x - x')$$

Multiply by $\psi_n^*(x)$ and integrate over V

$$\cancel{\sum_n} (\lambda - \lambda_n)\, a_n(x') = -4\pi\, \psi_n^*(x')$$

$$\Rightarrow \quad a_n(\vec{x}') = 4\pi \frac{\psi_n^*(\vec{x}')}{\lambda_n - \lambda}$$

$$\Rightarrow \quad \boxed{G(\vec{x}, \vec{x}') = 4\pi \sum_n \frac{\psi_n^*(x')\, \psi_n(x)}{\lambda_n - \lambda}}$$

Simple example free ~~partic~~ Green's function with boundaries only at ∞

$$(\nabla^2 + k^2)\, \psi_k(\vec{x}) = 0 \qquad \text{above form } f(\vec{x}) = 0,\ \lambda = 0$$

with continuum of eigenvalues k^2

$$\psi_k(x) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}$$

Normalized

$$\int d^3x\, \psi_{k'}^*(x)\, \psi_k(x) = \delta(k - k')$$

$$\Rightarrow \quad \sum_n \to \int d^3k$$

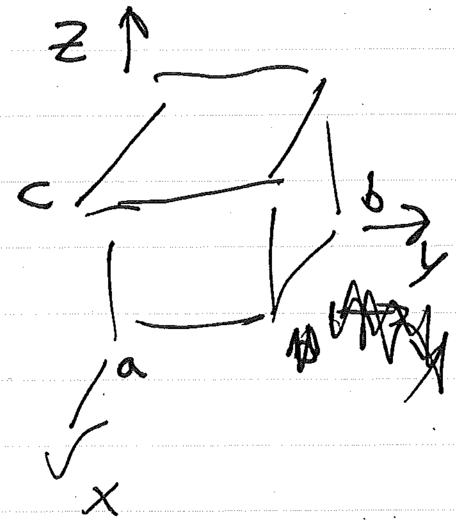
$$G(\vec{x}, \vec{x}') = 4\pi \int d^3k \frac{1}{(2\pi)^3} \frac{e^{ik\cdot(\vec{x} - \vec{x}')}}{k^2}$$

$$= \frac{1}{2\pi^2} \int d^3k \frac{e^{ik\cdot(\vec{x}-\vec{x}')}}{k^2} = \frac{1}{|\vec{x} - \vec{x}'|}$$

The F. Transform of

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{4\pi}{k^2}$$

Next consider a rectangular box



$$\left[\nabla^2 + k^2_{lmn}\right]\psi_{lmn}(x,y,z) = 0$$

$$\psi_{lmn} = \sqrt{\frac{8}{abc}}\,\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)$$

$$k^2_{lmn} = \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$

$$f(\vec{x}) = \lambda = 0$$

$$G(x,x') = \frac{32\pi}{abc}\sum_{lmn=1}^{\infty} \frac{\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)}{\pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)}$$

$$\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)$$

alternatively we could have solved in earlier way

$$G = \sum_{lm} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\psi_1(z_<)\psi_2(z_>)$$

Solution of Laplace eq.

$$A(z')\, e^{-k_{lm}z} + B(z')\, e^{+k_{lm}z} \qquad z > z'$$
$$A'(z')\, e^{-k_{lm}z} + B'(z')\, e^{+k_{lm}z} \qquad z < z'$$

Choose A, B to satisfy b.c. at $z' = c$

Choose A', B' to satisfy b.c. at $z' = 0$

$$\psi_1(z_<) \propto \sinh(k_{lm} z_<) \qquad \psi_1(0) = 0$$
$$\psi_2(z_>) \propto \sinh\left(k_{lm}(c - z_>)\right) \qquad \psi_2(c) = 0$$

$$k_{lm} = \pi^2 \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} \right)$$

Normalize to reproduce the delta function

$$G = \frac{16\pi}{ab} \sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right) \frac{\sinh(k_{lm} z_<)\, \sinh[k_{lm}(c - z_>)]}{k_{lm} \sinh[k_{lm} c]}$$

In one case you have a sum over eigenfunctions each of which satisfies both b.c. at $z=0$ and $z=c$.

In the other you have a single term

$$\psi_1(z_<)\ \psi_2(z_>)$$

where $\psi_1(0)=0$ but $\psi_1(c)\neq 0$

$\psi_2(c)=0$ but $\psi_2(0)\neq 0$

For the two approaches to be consistent

$$\frac{\sinh(K_{lm}z_<)\ \sinh[K_{lm}(c-z_<)]}{K_{lm}\sinh K_{lm}c} = \frac{2}{c}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi z'}{c}\ \sin\frac{n\pi z}{c}}{K_{lm}^2+\left(\frac{n\pi}{c}\right)^2}$$

Skip 3.13 about mixed b.c.
Start Reading Chap. 4
Multipoles, Electrostatics of Macroscopic Media, Dielectrics.

end 9/30

Pot. from a localized charge dist.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\ \frac{\rho(x')}{|\vec{x}-\vec{x}'|}$$