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Lecture # 13 Finite Square Well cont.

$$\kappa = \boxed{\frac{\sqrt{-2mE}}{\hbar}} \qquad l = \sqrt{2m(E+V_0)}/\hbar$$

assume $E < 0$ but $E + V_0 > 0$

Even solution

$$\psi(x) = \begin{cases} Fe^{-\kappa x} & x > a \\ D\cos lx & 0 < x < a \\ \psi(-x) & x < 0 \end{cases}$$

Cont. of ψ at $x = a$ $\qquad Fe^{-\kappa a} = D\cos la$

ψ' $\qquad -\kappa Fe^{-\kappa a} = -lD\sin la$

Need to match log derivatives $\frac{d\psi}{dx}/\psi$ at $x = a$

$$\boxed{\kappa = l\tan la}$$

This is an equation for the allowed energies E since $\kappa(E)$ and $l(E)$

Define $\quad z = la \qquad z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$

then $\quad \kappa a = \sqrt{z_0^2 - z^2}$

$$\tan z = \sqrt{(z_0/z)^2 - 1}$$

1) Wide deep well: If z_0 is large

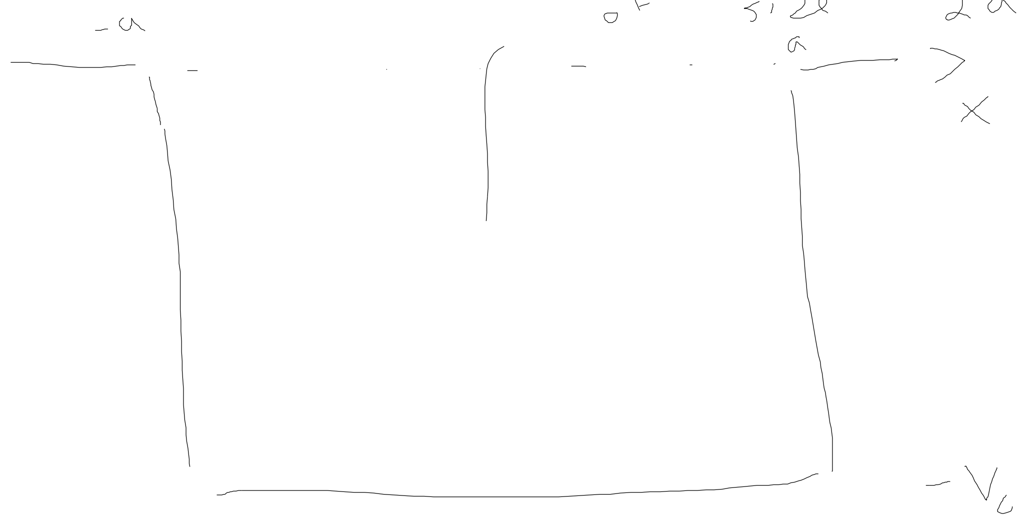
$\tan z \approx \infty$ thus $z \approx n\pi/2$, n odd,

$$E_n + V_0 = \frac{\hbar^2 l^2}{2m} = \frac{\hbar^2 \cancel{a^2} z^2}{2m a^2}$$

$$= \frac{\hbar^2}{2ma^2} n^2 \frac{\pi^2}{4} \qquad n \text{ odd}$$

$$\boxed{E_n + V_0 = \frac{\hbar^2}{2m} \frac{\pi^2 n^2}{(2a)^2}}$$

looks like an infinite square well of size $2a$



agrees with infinite square well limit

2) Shallow, narrow well

As $z_0 \to 0$, $z \to 0$ and $\tan z \approx z$

$$z^2 \approx z_0^2/z^2 - 1$$

$$z^4 + z^2 - z_0^2 = 0$$

$$z^2 = \frac{-1 \pm (1+4z_0^2)^{1/2}}{2} = \frac{(1+4z_0^2)^{1/2} - 1}{2}$$

$$z^2 \approx \frac{1 + 2z_0^2 - 16z_0^4/8 - 1}{2} = z_0^2 - z_0^4$$

$$z^2 \approx z_0^2 - z_0^4$$

$$E_n + V_0 = \frac{\hbar^2}{2ma^2}\left(z_0^2 - z_0^4\right)$$

$$z_0 = \frac{a}{\hbar}(2mV_0)^{1/2}$$

$$z_0^2 = \frac{a^2}{\hbar^2}\, 2mV_0$$

$$E_n + V_0 = \frac{\hbar^2}{2ma^2}\left[\frac{2mV_0 a^2}{\hbar^2} - \left(\frac{2mV_0 a^2}{\hbar^2}\right)^2\right]$$

$$E_n + V_0 = V_0 - \frac{2ma^2}{\hbar^2} V_0^2$$

$$\boxed{E_n \approx -\frac{2ma^2}{\hbar^2} V_0^2}$$

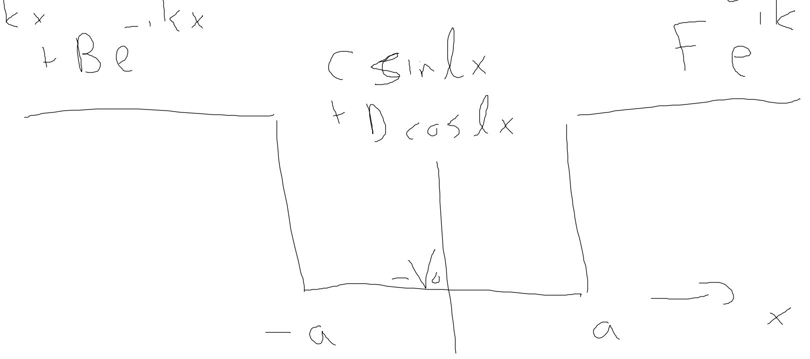
this energy is very small
Particle is just bound E is just below 0

Scattering states

If $E > 0$ then

$\Psi = Ae^{ikx} + Be^{-ikx}$

For $x < -a$



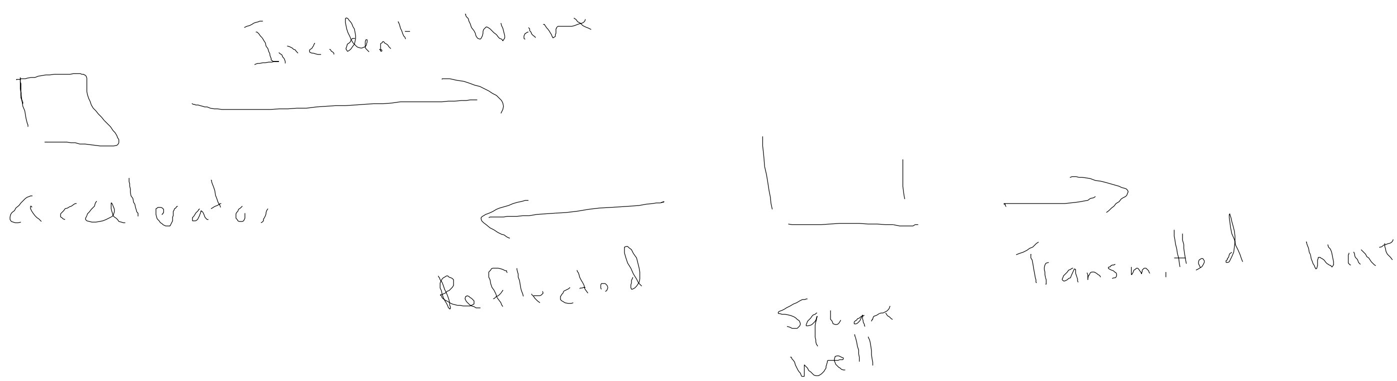
only a wave going to left

Ae^{ikx} = incident wave going to left (From accelerator)

Be^{-ikx} = reflected wave (particle bounces off of square well)

Fe^{ikx} = transmitted wave

No Ee^{-ikx} this would correspond to an incident wave from large x



Need to match wave function and its derivative at $x = a$ and $x = -a$

Gives four equations in 5 unknowns express B, C, D and F in terms of A

Example $Ae^{-ika} + Be^{ika} = C\sin(-la) + D\cos(-la)$

$C\sin la + D\cos la = Fe^{ika}$

Transmission coefficient

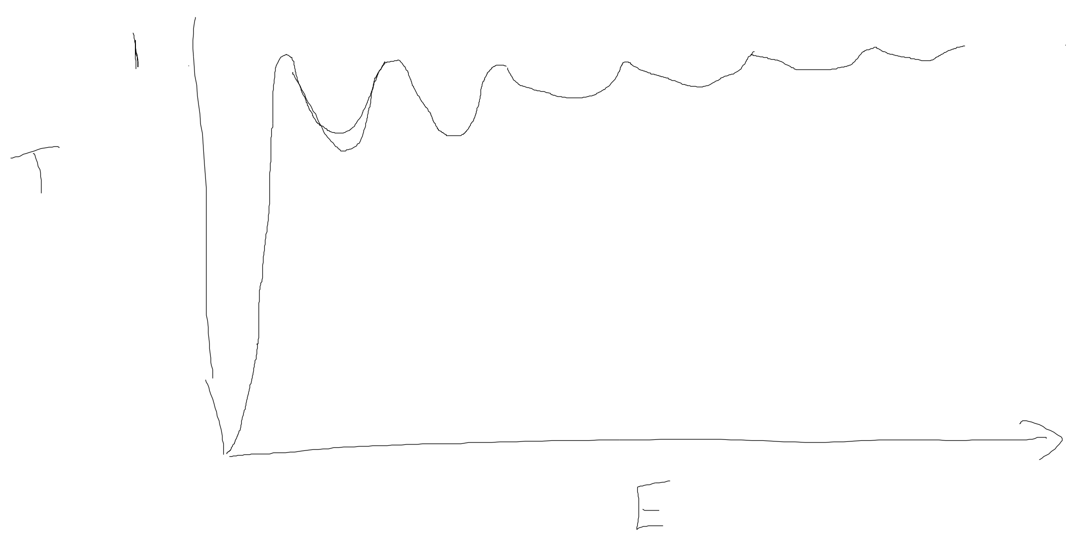
$$F = \frac{e^{-2ika} A}{\cos(2la) - i \frac{\sin(2la)}{2kl}(k^2 + l^2)}$$

see problem 2.31

The fraction of the prob. transmitted is

$$|F|^2 / |A|^2 = T$$

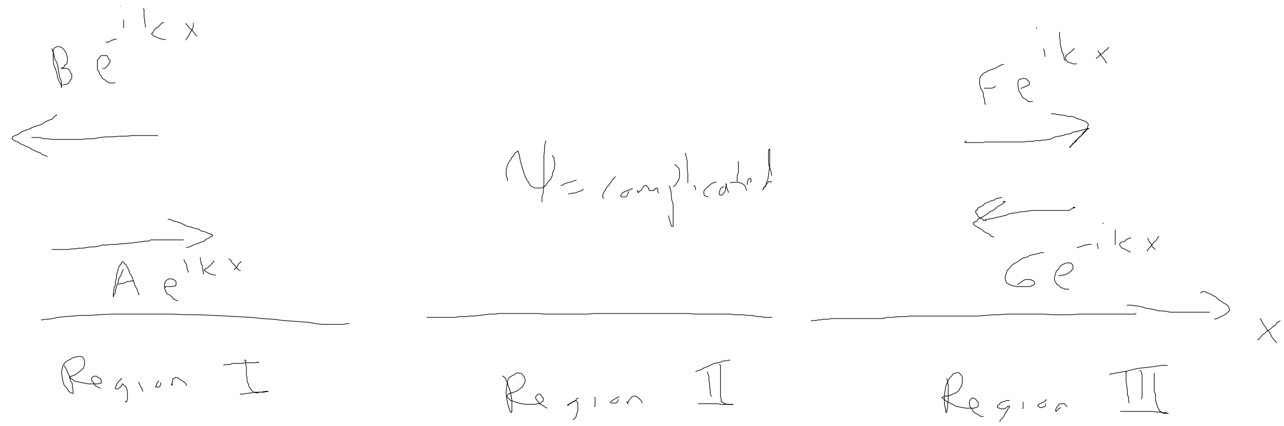
$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$$



Note $V = -V_0$ is an attractive pot.
Classically particle would speed up
and then slow back to its original
velocity but $T \equiv 1$

Scattering Matrix

Consider a localized pot which is only $\neq 0$ in region II



Incident waves A, G produce reflected waves B, F

$$\begin{pmatrix} B \\ F \end{pmatrix} = \left[\underline{\underline{S}} \right] \begin{bmatrix} A \\ G \end{bmatrix}$$

$\left[\underline{\underline{S}} \right]$ = two x two complex matrix found by matching wave functions and derivatives at boundaries of Region II

$$\left[\underline{\underline{S}} \right] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

Note $\underline{\underline{S}}$ is a very general concept
The scattering matrix exists for every
pot. (short ranged).

Example: reflection coef $\left|\dfrac{|B|^2}{|A|^2}\right|$ $G=0$

with no wave incident from ~~left~~ right $G=0$

$$R_l = \frac{|B|^2}{|A|^2} = |S_{11}|^2 \qquad T_l = \left|\frac{|F|^2}{|A|^2}\right| = |S_{21}|^2$$

$G=0$

For scattering from right $A=0$

$$R_r = \left|\frac{|F|^2}{|G|^2}\right| = |S_{22}|^2 \qquad T_r = \frac{|B|^2}{|G|^2} = |S_{12}|^2$$

$A=0$

$$R_l + T_l = 1 \Rightarrow |S_{11}|^2 + |S_{21}|^2 = 1$$

likewise $|S_{22}|^2 + |S_{12}|^2 = 1 = R_r + T_r$

$\underline{\underline{S}}$ is unitary