Lecture #12: Change of basis, normalization

Can represent state of a particle with a coordinate space wave function.

\[ \psi(x) \psi(x) \, dx = \text{Prob to be between } x \text{ and } x+dx \]

Normalization

\[ \int_{-\infty}^{\infty} \psi^* \psi \, dx = 1 \]

or could expand in Energy eigenstates:

\[ \psi(x) = \sum c_i \psi_i(x) \]

Where the \( \psi_i \) are orthogonal:

\[ \int_{-\infty}^{\infty} \psi_i^* \psi_j \, dx = \delta_{ij} \]

Thus

\[ \int_{-\infty}^{\infty} \psi^* \psi \, dx = \sum c_i^* \sum c_j \]

\[ \int_{-\infty}^{\infty} \psi_i^* \psi_j \, dx = \sum c_i^* c_j \delta_{ij} \]

\[ = \sum c_i^* c_i = 1 \]
can think about representing state of particle with the set

\[ \psi \in \mathbb{C} \]

When \( \psi \in \mathbb{C} \), prob. to be in state \( i \)

Clearly these \( c_i \) are normalized \( \sum_i |c_i|^2 = 1 \)

Alternatively can use a momentum space wave function \( \phi(k) \).

Normalization \( \int_{-\infty}^{\infty} \phi(k) \phi^*(k) dk = 1 \) (will show)

Prob. interp. \( \phi(k) \phi^*(k) dk \) to find particle with momentum between \( h(k) \) and \( h(k+dk) \)

Remark: Free particle

\[ \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{-ik\cdot x} \phi(k) \]

This satisfies eq. for \( V=0 \) for any \( \phi(k) \).

\[ \psi^*(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk' e^{-ik'\cdot x} \phi^*(k') \]
Compute normalization:

\[
\int_{-\infty}^{\infty} \Psi(x) \Psi^*(x) \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(k) \phi(k') e^{-i k' x} \, dk \, dk' = 1
\]

Do integral first:

\[
\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{i (k-k') x} \, dx = \delta(k-k')
\]

\[
\int_{-\infty}^{\infty} \phi(x) \phi^*(x) \, dx = \int_{-\infty}^{\infty} \phi(k) \phi(k) \, dk = \int_{-\infty}^{\infty} \phi(k) \phi(k) \, dk = 1
\]

Normalization integral locks the same in x space as in k space.

So we have a normalized wave function with a probability interpretation and we can expand the wave function in any convenient basis.
Finite Square Well

\[ V = \begin{cases} \ -V_0, & -a < x < a \\ \ 0, & |x| > a \end{cases} \]

with \( V_0 > 0 \)

\[ V(x) \]

For \( |x| > a \), \( V = 0 \)

\[-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi \quad \text{in} \quad x = 0, \quad \frac{d^2 \psi}{dx^2} = \frac{\hbar^2}{2m} \psi \]

\[ k = \sqrt{\frac{2mE}{\hbar}} \]

Look for bound states \( E < 0 \)

\[ \psi = \begin{cases} \ B e^{i k x} & x < -a \\ F e^{-i k x} & x > a \end{cases} \]

Want \( \psi \to 0 \) as \( |x| \to \infty \)
For $|x| < a$

\[- \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} - V_0 \psi = E \psi\]

\[0 \quad \frac{\partial^2 \psi}{\partial x^2} = - \ell^2 \psi\]

\[\ell = \sqrt{\frac{2m}{h^2}} \left( E + V_0 \right)\]

Want $E + V_0 > 0$ with $E < 0$

$|E| < V_0$, binding $E$ must be less than $V_0$

$\psi = C \sin(\ell x) + D \cos(\ell x) \quad |x| < a$

Expect $\psi$ to be either even

$\psi(x) = \psi(-x)$

or odd

$\psi(x) = -\psi(-x)$ since $\ell > 0$ is even $V(x) = V(-x)$

Look for even solutions $C = 0$

$\psi = \left\{ \begin{array}{ll} F e^{-\gamma x} & x > a \\ 0 \cos \ell x & 0 < x < a \\ \psi(-x) & x < 0 \end{array} \right.$
Wave function must be cont. at \( x = a \)

\[
F - e^{-\frac{1}{2}a} = D \cos \lambda a
\]

also \( \frac{dF}{dx} \) must be cont. (note \( V \)

is finite \( \frac{dF}{dx} \))

\[
- \frac{\pi}{2}Fe^{-\frac{1}{2}a} = -\pi D \sin \lambda a
\]

or

\[
\lambda \pi = \frac{\pi}{2} \tan \lambda a
\]

This is an eigenvalue equation for the allowed \( E \)

\[
\lambda \pi = \frac{\sqrt{2mE}}{\hbar}
\]

\[
\lambda = \frac{\sqrt{2m(E+V_0)}}{\hbar}
\]

set

\[
Z = \lambda a \quad \text{and} \quad Z_0 = \frac{a}{\hbar} \sqrt{2mV_0}
\]

\[
1 + \lambda^2 = \frac{2mV_0}{\hbar^2}
\]

\[
1 + \lambda^2 = \frac{Z_0^2}{Z^2} - Z^2
\]

\[
\sqrt{Z_0^2 - Z^2} = Z + \tan Z
\]

or

\[
\tan Z = \sqrt{(Z_0/Z)^2 - 1}
\]

Can solve numerically
Wide deep well \( Z_0 \rightarrow \infty \)

\[ \tan Z = \infty \Rightarrow \cos Z = 0 \]

\[ Z_n \approx n \frac{\pi}{2} \quad n \text{ odd} \]

\[ E_n + V_0 \approx \frac{n^2 \frac{\hbar^2}{2m} + \frac{h^2}{2m}}{(2a)^2} \]

Shallow, narrow well.

Let \( Z_0 \rightarrow 0 \) expect \( Z \rightarrow Z_0 \rightarrow 0 \)

\[ \tan Z_0 = Z = \left( \frac{Z_0^2}{Z^2} \right)^{\frac{1}{2}} \]

\[ Z^2 = \frac{Z_0^2}{Z^2} - 1 \]

\[ a^2 + Z^2 = 2 \]

\[ Z^2 - Z_0^2 < 0 \]

\[ Z = -1 \pm \left( 1 + 4Z_0^2 \right)^{\frac{1}{2}} \]

\[ Z^2 = \frac{(1 + 4Z_0^2)^{\frac{1}{2}}}{2} \]

\[ Z \approx Z_0 \Rightarrow E \approx 0 \]

In one him an attractive pot.

no matte how weak always has
at least one bound state.