Lecture # 11 Delta functions

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

and

$$\int dx\, \delta(x) = 1$$

Example:

$$\delta(x) = \begin{cases} 1/2a & |x| < a \\ 0 & |x| > a \end{cases}$$

in limit $a \to 0$

Claim

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx}$$

This is an ill defined integral for $x \neq 0$ since it just keeps oscillating but it can be defined by for example letting integral run from $-L$ to L and take limit as $L \to \infty$

Last time: Free particle

$$\psi(\cancel{x}) = \frac{1}{\sqrt{2\pi}} \int dk\, \phi(k)\, e^{ikx}$$

This solves S. eq. for any $\phi(k)$

$$\psi^*(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk' \, \phi^*(k') \, e^{-ik'x}$$

Compute normalization integral

$$\int_{-\infty}^{\infty} dx \, \psi^*(x) \, \psi(x) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dk \; \underbrace{\frac{1}{\sqrt{2\pi}} \phi^*(k') e^{-ik'x}}_{\psi^*(x)} \; \underbrace{\frac{1}{\sqrt{2\pi}} \phi(k) e^{ikx}}_{\psi(x)}$$

Do integral over x first

$$\int_{-\infty}^{\infty} dx \, \frac{1}{2\pi} e^{ix(k-k')} = \delta(k-k')$$

$$\int dx \, \psi^*(x) \, \psi(x) = \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dk \; \delta(k-k') \, \phi^*(k') \, \phi(k)$$

Now do integral over k'

$$\int_{-\infty}^{\infty} dk' \, \delta(k-k') \, \phi^*(k') = \phi^*(k)$$

$$\boxed{\int_{-\infty}^{\infty} dx \, \psi^*(x) \psi(x) = \int_{-\infty}^{\infty} dk \; \phi^*(k) \, \phi(k)}$$

Normalization in momentum (or k) space

has same form as normalization in coordinate space.

(a) $\phi(k)$ is momentum space wave function.

(b) $\phi^*(k)\,\phi(k)$ is probability density.

Prob to find particle with momentum between $\hbar k$ and $\hbar(k+dk)$ is

(c) $P = \phi^*(k)\,\phi(k)\,dk$

Can have any normalizable function $\phi(k)$ to describe an allowed free particle wave function which satisfies Schrodinger equation. Note, it is not a stationary state since it has a range of different ks.

Time dependence is nontrivial

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k)\, e^{i\left[kx - \frac{\hbar k^2}{2m}t\right]}\,dk$$

since it is not simply one overall phase $e^{-iEt/\hbar}$

Delta function potential

The deuteron is a bound state of a neutron and a proton. It is bound by the strong but very short range nuclear force.

The range of the force is small compared to the size of the bound state. Approximate interaction by a delta function

$$\boxed{V(x) = -\alpha\,\delta(x)}$$

α is a parameter that describes the volume integral of the potential

$$\int dx\, V(x) = -\alpha$$

The negative sign says pot. is attractive (assume $\alpha > 0$) to give binding

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\psi(x) = E\,\psi(x)$$

Everywhere except $x = 0$ can ignore $\delta(x)$

$$\frac{d^2\psi}{dx^2} = -\left(\frac{2mE}{\hbar^2}\right)\psi(x)$$

Now we search for a bound state

$$E < 0$$

For example deuteron is bound by 2.2 MeV. For deuteron $E = -2.2$ MeV I.E. it takes a 2.2 MeV γ ray to seperate the n from the p.

$$\kappa = \sqrt{\frac{-2mE}{\hbar^2}} \Rightarrow \psi = e^{\pm \kappa x}$$

κ is real since $E < 0$

$$\psi(x) = \begin{cases} B e^{\kappa x} & \text{for } x < 0 \\ B e^{-\kappa x} & \text{for } x > 0 \end{cases}$$

Chose sign of exp. so wave function is normalizable $\psi(x) \to 0$ as $|x| \to \infty$

What about at $x = 0$?
~~Wave~~ delta function makes itself felt through boundary conditions

(1) Wave function $\psi(x)$ is always continuous
(2) $d\psi/dx$ is continuous except

where pot. is infinite.

Integrate schrodinger eq. in a small interval from $-\epsilon$ to ϵ. Then take limit $\epsilon \to 0$

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon} dx \, \frac{d^2\psi}{dx^2} - \alpha\int_{-\epsilon}^{\epsilon} dx \, \delta(x)\psi(x)$$

$$= \int_{-\epsilon}^{\epsilon} dx \, E\,\psi(x)$$

$$-\frac{\hbar^2}{2m}\left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} - \alpha\,\psi(0) = \cancelto{0}{E\,\psi(0)\,2\epsilon}$$

as $\epsilon \to 0$

Thus
$$\boxed{\left.\frac{d\psi}{dx}\right|_{x=0+\epsilon} - \left.\frac{d\psi}{dx}\right|_{x=0-\epsilon} = -\left(\frac{2m\alpha}{\hbar^2}\right)\psi(0)}$$

This is boundary condition for delta function potential

$$\psi = B e^{\kappa x} \qquad x < 0$$

$$\left.\frac{d\psi}{dx}\right|_{x=-\epsilon} = \kappa B$$

$$\psi = B e^{-\kappa x} \qquad x > 0$$

$$\left.\frac{d\psi}{dx}\right|_{x=+\epsilon} = -\kappa B$$

$$-\kappa B - (+\kappa B) = -\left(\frac{2m\alpha}{\hbar^2}\right) B$$

$$\left.\frac{d\psi}{dx}\right|_{+\epsilon} - \left.\frac{d\psi}{dx}\right|_{-\epsilon} \qquad \psi(0) = B$$

$$2\kappa = \frac{2m\alpha}{\hbar^2}$$

$$\Rightarrow \kappa = \frac{m\alpha}{\hbar^2} \qquad E = -\frac{\hbar^2 \kappa^2}{2m}$$

$$\boxed{E = -\frac{\hbar^2 m^2 \alpha^2}{2m\hbar^4} = -\frac{m\alpha^2}{2\hbar^2}}$$

Binding Energy depends on square of strength of pot.

$$\psi = B e^{-m\alpha|x|/\hbar^2}$$

Normalize $\int dx\, \psi^* \psi = 1$

$$\boxed{\psi = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}}$$