Lecture #10 The free particle

Last time

$$\psi_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi)\, e^{-\xi^2/2}$$

$H_n(\xi)$ = Hermite polynomial $\begin{cases} H_0 = 1 \\ H_1 = 2\xi \\ H_{n+1} = 2\xi H_n - 2nH_{n-1} \end{cases}$

For small n look nothing like classical motion.

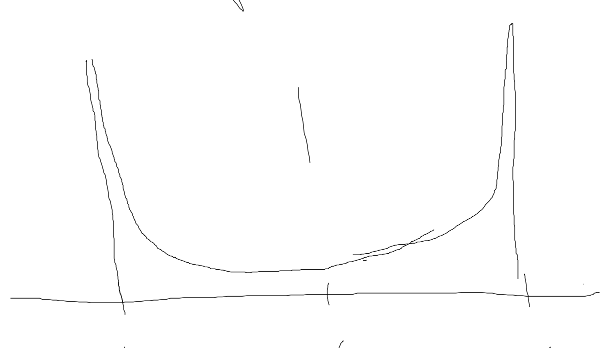
Classical H. Osc

Turning points $E = \frac{1}{2} k x_t^2$

so

$$(n+\tfrac{1}{2})\hbar\omega = \tfrac{1}{2}k\, x_t^2$$

$$x_t = \pm\sqrt{2(n+\tfrac{1}{2})\hbar\omega/k}$$



For quantum system

(a) Some prob. to be found outside classical turning points

(b) In limit $n \to \infty$ then $\psi_n^* \psi_n$ comes close to classical prob. distribution

<u>Correspondance principle</u>

Classical behavior recovered in limit of large quantum numbers

<u>Free particle</u> $V = 0$

Seems like simplest case

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\psi = Ae^{ik \cdot x} + Be^{-ikx} \qquad \frac{d^2\psi}{dx^2} = -k^2\psi$$

There are no boundary conditions to restrict allowed k or E.

$\Rightarrow$ Even in QM, a free particle can have any energy add time dependence $e^{-iEt/\hbar}$

$$\Psi(x,t) = A e^{ik\left(x - \frac{\hbar k}{2m}t\right)} + B e^{-ik\left(x + \frac{\hbar k}{2m}t\right)}$$

Two traveling waves $k > 0$ goes to right
$k < 0$ (B) goes to left

Argument of traveling wave $x - \cancel{v}\, vt$

$$V_{quantum} = \frac{\hbar k}{2m} = \frac{p}{2m} = \frac{1}{2} V_{classical}$$

Note $p = \hbar k$ for these waves

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad \hat{p}\left(Ae^{ikx}\right) = \hbar k \left(Ae^{ikx}\right)$$

What's strange?? with factor of $\frac{1}{2}$?

First big problem. <u>Can't normalize</u>

$$\int_{-\infty}^{\infty} dx\, \psi^* \psi = \infty$$

A pure plane wave state $e^{\pm ikx}$
plays no <u>physical role</u>
but it does play a very useful
<u>mathematical</u> role!

Can't have a free particle with

a definite energy. Instead we will build normalizable wave packets by superposing a range of k values and thus a spread in $E = \frac{\hbar^2 k^2}{2m}$

Replace A, B by $\int \frac{1}{\sqrt{2\pi}} \phi(k)\, dk$

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \phi(k)\, e^{i\left[kx - \frac{\hbar k^2}{2m} t\right]}}$$

- Most general superposition of plane waves of different momentum ($\hbar k$). Factor of $\frac{1}{\sqrt{2\pi}}$ is just a convention.
- Can think of $\phi(k)$ as momentum space wave function $\phi^*(k)\,\phi(k)$ Prob. density to find particle with momentum $\hbar k$.

Fourier transform

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \phi(k)\, e^{ikx}$$

in general $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx}$

Fourier transform can be <u>inverted</u>

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

Note e^{-ikx} for inverse and e^{+ikx} for (direct) Fourier transform.

Include $1/\sqrt{2\pi}$ so transforms would be normalized

$$f(x) = \frac{1}{\sqrt{2\pi}} \int dk \, F(k) \, e^{ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' \, e^{ikx} \, e^{-ikx'} f(x')$$

Use

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' \, e^{-ikx'} f(x')$$

$$f(x) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, e^{ik(x-x')} f(x')$$

We will come back to meaning of this

Normalization

$$\int_{-\infty}^{\infty} dx\, \Psi^*(x,0)\,\Psi(x,0) = \int_{-\infty}^{\infty} dx\, \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk'\, e^{-ik'x}\, \phi^*(k')}_{\Psi^*(x,0)}\ \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, e^{ikx}\, \phi(k)}_{\Psi(x,0)}$$

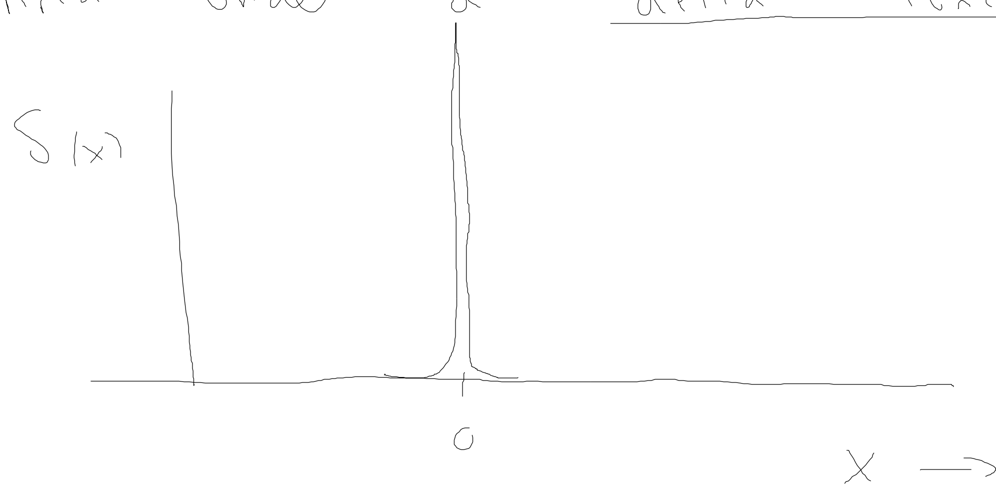
$$= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk'\ \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{ix(k-k')}\ \phi^*(k')\,\phi(k)$$

The "Function" $\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-x')} \equiv \delta(x-x')$

$$\delta(x) \equiv \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

Such that $\int dx\, \delta(x) = 1$

Area under a <u>delta function</u> is one



The most important use of a delta function is

$$\int dx\ \delta(x)\ f(x) \equiv f(0)$$

$$\int_a^b dx\ \delta(x-x_0)\ f(x) \equiv f(x_0)$$

For any function $f(x)$. The above is true if

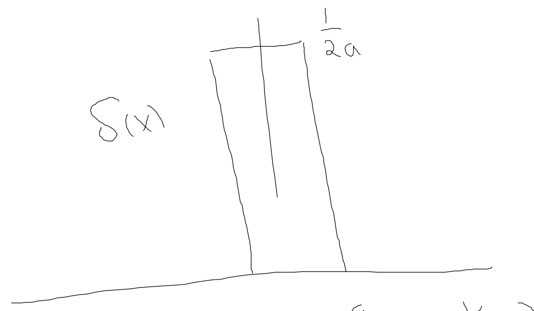
$$a < x_0 < b$$

else integral is zero if x_0 is outside range of integration.

Example
$$\delta(x) = \begin{cases} 1/2a & \text{if } |x| < a \\ 0 & \text{else} \end{cases}$$

and then take limit $a \to 0$

$$\int_{-a}^{a} dx\ \frac{1}{2a} = 1 \quad \checkmark$$



Claim

$$\boxed{\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}}$$