Lecture #9 Harmonic Osc. Cont.

\[-\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + V(x) \Psi = E \Psi\]

\[V = \frac{1}{2} k x^2 \quad \omega = \sqrt{\frac{k}{m}}\]

Dimensionless Variables

\[\xi = \sqrt{\frac{m \omega}{\hbar}} \times K = \frac{E}{\left(\frac{1}{2} m \omega\right)}\]

\[\frac{d^2 \Psi}{d\xi^2} = \left(\xi^2 - K\right) \Psi(\xi)\]

Last time

(a) Pull out asymmetric behavior \( e^{\pm \xi^2/2}\)

Choose \( e^{-\xi^2/2} \) to be normalizable

(b) Guess \( \Psi(\xi) = h(\xi) e^{-\xi^2/2}\)

\[h'' - 2\xi h' + (K - 1) h = 0\]

(c) Expand \( h = \sum a_i \xi^i\)

\[a_{i+2} = \frac{2i + 1 - K}{(i+1)(i+2)} a_i\]
(d) Impose b.c. $h(t)$ can’t blow up to much as $t \to \infty$ otherwise $\psi(t)$ will not be normalizable.

$\Rightarrow$ Series must terminate

$$K = 2n+1 \quad n=0,1,2,...$$

$$E = K \frac{1}{2} \hbar \omega = (n+\frac{1}{2}) \hbar \omega$$

With this choice of $K$ and all higher $n \geq 0$

Numerical solution to see how b.c. determines allowed $E$

$$\frac{d\psi}{dx} = \frac{1}{2h} \left( \psi(x+h) - \psi(x-h) \right)$$

$$\frac{d^2\psi}{dx^2} = \frac{1}{\hbar^2} \left[ \psi(x+h) + \psi(x-h) - 2\psi(x) \right]$$

$0 \quad h \quad 2h \quad 3h \quad 4h \quad x \to$

Calculate $\psi$ on grid $\psi_i = \psi(ih)$
can see formula for 2nd der. as follows

\[ \frac{d^2 \psi}{dx^2} = \frac{1}{h} \left[ \frac{d\psi}{dx} \bigg|_{x + \frac{h}{2}} - \frac{d\psi}{dx} \bigg|_{x - \frac{h}{2}} \right] \]

and \( \frac{d\psi}{dx} \bigg| = \frac{1}{h} \left( \psi(x+h) - \psi(x) \right) \)

\[ \frac{d\psi}{dx} \bigg| = \frac{1}{h} \left[ \psi(x) - \psi(x-h) \right] \]

\[ \psi'' = (x^2 - K) \psi(x) \]

\[ \frac{1}{h^2} \left( \psi_{i+1} + \psi_{i-1} - 2\psi_i \right) = (x_i^2 - K) \psi_i \]

Solve for \( \psi_{i+1} \)

\[ \psi_{i+1} = h^2 (x_i^2 - K) \psi_i + 2\psi_i - \psi_{i-1} \]

Consider odd solution these have

\[ \psi(0) = 0 \quad \Rightarrow \quad \psi_0 = 0 \]

Guess \( \psi_1 = \epsilon \quad \epsilon = \text{any small} \text{ number} \)

go back and normalize later
\[ \Psi_2 = h^2 (x^2 - K) \psi + 2 \psi = 0 \]

etc. then set \( \Psi_3 \) from \( \Psi_2 \) and \( \Psi_1 \)

Procedure:
(a) Guess \( K \)
(b) Integrate out to large \( x \)
(c) Repeat and adjust \( K \)
so \( \Psi(x) \rightarrow 0 \) as \( x \rightarrow \infty \) to be normalizable.

Basic code: HSC, bas See course web site for C code programs in course lectures.

\[
\Psi_{i+1} = h^2 (x^2 - K) \Psi_i + 2 \Psi_i \\
\Psi_{i} = \Psi_{i+1}, \quad \Psi_{i-1} = -\Psi_{i}
\]

Shooting method: adjust \( K \) till wave function behaves as \( x \rightarrow \infty \)

\[ E = (n + \frac{1}{2}) \frac{\hbar^2}{2m} \] with \( n \) odd

Note: \( n \) even does not have \( \Psi(\infty) = 0 \)
Example of power series

\[ n = 2 \Rightarrow k = 5 \]

\[ a_{2} = \frac{18 - 5}{1 \cdot 2} \]
\[ a_{2} = -2a_{0} \]

\[ a_{4} = \frac{5 - 5}{3 \cdot 4} a_{2} = 0 = a_{6} \ldots \]

\[ \Psi_{2} (\xi) = a_{0} (1 - 2 \xi^{2}) e^{-\frac{\xi^{2}}{2}} \]

Can find \( a_{0} \) by normalization.

Finite polynomials \( h(\xi) \) are related to Hermite polynomials \( H_{n}(x) \).

\( H_{n}(x) \) is polynomial of order \( n \) with 1st root \( 2^{n} \)

\( H_{0}(x) = 1, \quad H_{1}(x) = 2x, \quad H_{2}(x) = 4x^{2} - 2 \)

\( H_{3}(x) = 8x^{3} - 12x \quad \ldots \)

\[ N \approx 6, \quad H_{n}(x) = 2^{n} H_{n}(x) - 2n H_{n-1}(x) \]

\[ \Psi_{2}(\xi) = \left( \frac{a_{0}}{-2} \right) H_{2}(\xi) e^{-\frac{\xi^{2}}{2}} \]
Use

in wave functions. 

Harmonic oscillator.

is the correspondence principle. 

\[ E = \frac{1}{2} m \omega^2 x^2 \]

\[ x = \sqrt{\frac{2(E)}{m \omega^2}} \]

\[ \psi_n(x) = \frac{1}{(\pi a_0^2)^{1/4}} e^{-x^2/2a_0^2} \]

\[ \langle x \rangle = \int x |\psi_n(x)|^2 dx \]

\[ N \text{ normalized harmonic wave function} \]

Normalization integral can be done.