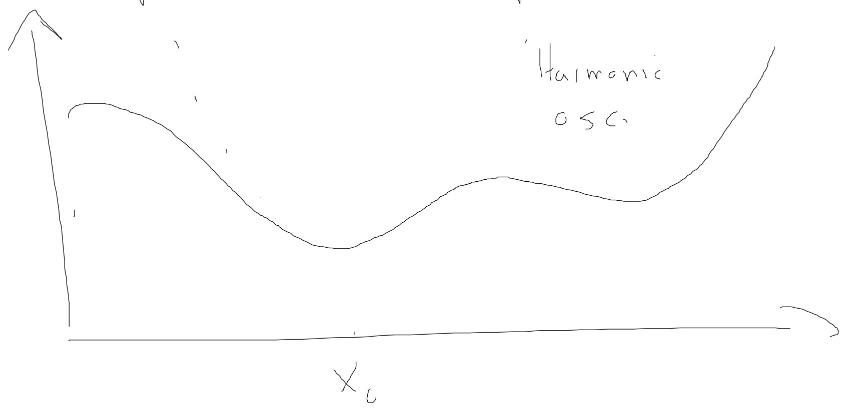
Lecture #8 Harmonic Osc.

Taylor expand any pot. about a minimum



$$V(x) \simeq V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2}V''(x_0)(x-x_0)^2 + \dots$$

at a minimum $V'=0$. Chose zero of energy $V(x_0)$ and set origin so $x_0 = 0$

$$\boxed{V \simeq \frac{1}{2} k x^2} \qquad k = \text{const.}$$

Any small osc. about an equilibrium position look harmonic.

Classical freq. $\omega = \sqrt{\frac{k}{m}}$

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \psi = E\psi}$$

$$V = \tfrac{1}{2} k x^2 = \tfrac{1}{2} m\omega^2 x^2$$

Algebraic solution is "too clever"

Clean up eq. a little

$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$ greek xi

$$\boxed{\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi} \quad (*)$$

$$K = E/(\tfrac{1}{2}\hbar\omega) = \frac{2E}{\hbar\omega}$$

ξ, K are dimensionless

Solve eq. $(*)$ by power series

(i) General (ugly) way to solve dif. equations

(ii) Pull out asym. behavior first

(iii) Expand remainder in a power series

(iv) Impose b.c. to get energy eigenvalues c, allowed values of K

(ii) For large ξ

$$\frac{\partial^2\psi}{\partial\xi^2} \simeq \xi^2\psi$$

$$\psi = A e^{-\xi^2/2} + \cancel{B} e^{\xi^2/2}$$

bad not normalizable

(iii) Guess $\psi(\xi) = h(\xi)\, e^{-\xi^2/2}$

For general $h(\xi)$ this can still be exact.

$$\frac{\partial \psi}{\partial \xi} = \left[\frac{\partial h}{\partial \xi} - \xi\right] e^{-\xi^2/2}$$

$$\frac{\partial^2 \psi}{\partial \xi^2} = \left[\frac{\partial^2 h}{\partial \xi^2} - 2\xi \frac{\partial h}{\partial \xi} + h(\xi^2 - 1)\right] e^{-\xi^2/2}$$

So $$\frac{\partial^2 \psi}{\partial \xi^2} = (\xi^2 - K)\psi$$

$$\left[\frac{\partial^2 h}{\partial \xi^2} - 2\xi \frac{\partial h}{\partial \xi} + (K-1)h\right] = 0$$ ξ^2 canceled

$**$

Expand $$h = \sum_{j=0}^{\infty} a_j \xi^j$$

$$h' = \sum_{j=0}^{\infty} j\, a_j\, \xi^{j-1}$$

$$\frac{\partial^2 h}{\partial \xi^2} = h'' = \sum_{j=0} j(j-1)\, a_j\, \xi^{j-2}$$

Change dummy variables for h''

$$h'' = \sum_{i=0} (i+2)(i+1)\, a_{i+2}\, \xi^i \qquad i = j-2$$

$$\sum_i \left[(i+2)(i+1)\, a_{i+2}\xi^i - 2i\, a_i\, \xi^i + (K-1)a_i\, \xi^i\right] = 0$$

If true for all ξ then each coef vanishes

$$(i+2)(i+1)a_{i+2} + \cancel{2i}\left[K-2i-1\right]a_i = 0$$

True for all i

$$\boxed{a_{i+2} = \frac{2i+1-K}{(i+2)(i+1)}\,a_i}$$

Recursion formula for a_{i+2} given a_i

given a_0 get $a_2, a_4 \cdots a_{even}$

given arbitrary a_1 get $a_3, a_5 \cdots a_{odd}$

Two arbitrary constants a_0, a_1 for 2nd order dif. eq. then get all other coef.

For large i

$$a_{i+2} \approx \frac{2}{i}\,a_i$$

$$a_i \approx C \Big/ \left(\frac{i}{2}\right)!$$

$$h(\xi) = C\sum_i \frac{1}{(i/2)!}\,\xi^i \approx C e^{\xi^2}$$
$$= C\sum_k \frac{1}{k!}\,\xi^{2k}$$

If series does not terminate will blow up as e^{ξ^2}

$$\psi(\xi) = h(\xi)e^{-\xi^2/2} \longrightarrow e^{\xi^2/2}$$

gives back asym. behavior we don't want

Solution to this problem is to chose K so that there is some

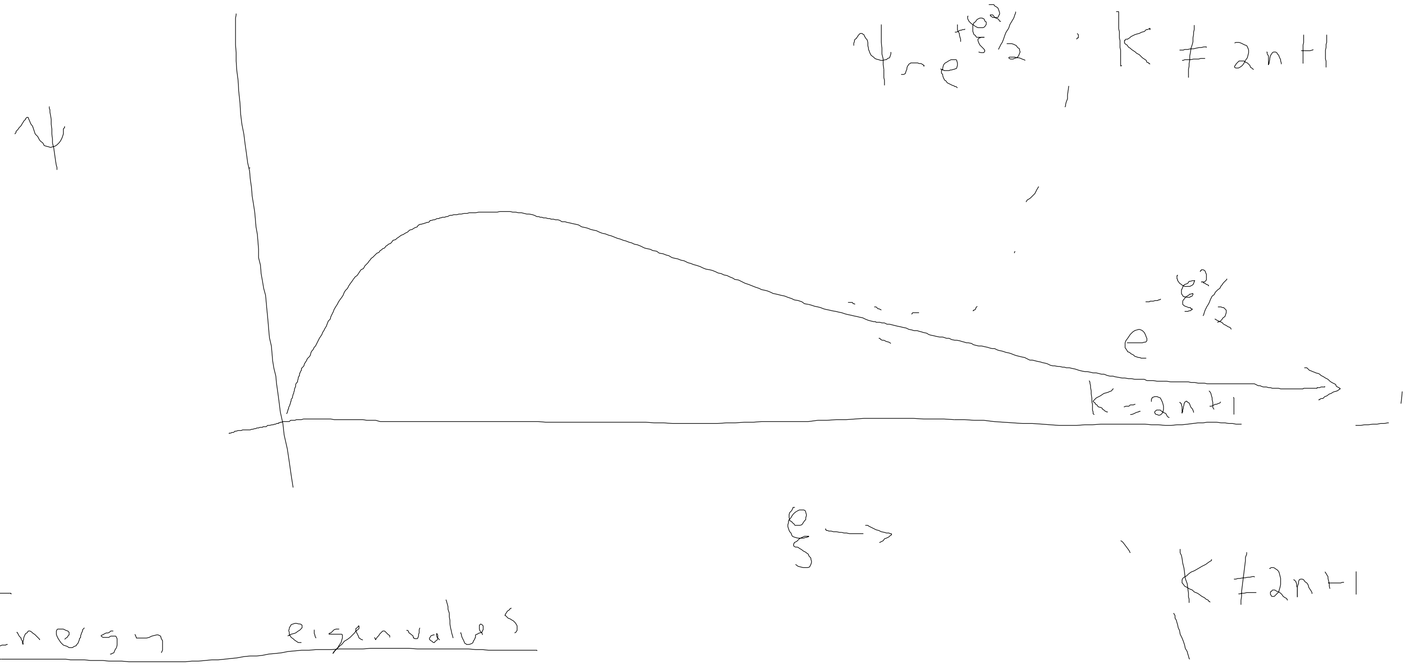
$$i = n$$

such that

$$2n+1 = K$$

then $a_i \equiv 0$ for $i \geq n$

because $a_{i+2} = \dfrac{2i+1-K}{(i+2)(i+1)}\, a_i$



Energy eigenvalues

$$K = 2n+1$$

$$\boxed{E = \tfrac{1}{2}\hbar\omega K = \left(n+\tfrac{1}{2}\right)\hbar\omega}$$

$n =$ integer $0, 1, 2, \ldots$

Ground state $n = 0 \Rightarrow K = 1$

$$a_2 = \frac{2\cdot 0 + 1 - 1}{(2+0)(1+0)} a_0 = 0$$

$$a_4 = \frac{3-1}{4\cdot 3} a_2 = \frac{2}{12} a_2 = 0 = a_6 \cdots$$

$h(\xi) = a_0$

$$\boxed{\psi_0 = a_0 e^{-\xi^2/2}}$$ ground state

$$\xi = \left(\frac{m\omega}{\hbar}\right)^{1/2} x$$

$$\psi_0 = a_0 e^{-\left(\frac{m\omega}{\hbar}\right)\frac{x^2}{2}}$$

$$\boxed{\psi_0(x) = a_0 e^{-\frac{\sqrt{mK}}{\hbar}\frac{x^2}{2}}}$$ $E = \frac{\hbar\omega}{2}$

Determine a_0 by normalization

$$\int_{-\infty}^{\infty} \psi_0^*(x)\, \psi_0(x)\, dx = 1$$

For $n = 1$ chose $a_0 \equiv 0$ then $K = 2n+1$

$$a_3 = \frac{[(2\cdot 1 + 1) - (2n+1)]\, a_1}{3\cdot 2} = 0 = a_5 \cdots$$

$h(\xi) = a_1 \xi$

$$\psi_1 = a_1 \xi e^{-\xi^2/2}$$ $E = \frac{3}{2}\hbar\omega$

$$\psi_2 = a_0 (1 - 2\xi^2)\, e^{-\xi^2/2}$$ $E = \frac{5}{2}\hbar\omega$