Lecture #8  Harmonic Osc.

Taylor expand any pot. about a minimum

\[ V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2} V''(x_0)(x-x_0)^2 + \ldots \]

at a minimum \( V' = 0 \). Choose zero of energy \( V(x_0) \) and set origin so \( x_0 = 0 \)

\[ V = \frac{1}{2} k x^2 \]

Any small position lock osc. about an equilibrium.

Classical freq. \( \omega = \sqrt{\frac{k}{m}} \)

\[ -\frac{k}{2m} \frac{\partial^2 Y}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 Y = E Y \]

\[ V = \frac{1}{2} k x^2 = \frac{1}{2} m \omega^2 x^2 \]

Algebraic solution is "to clever."
\[ \psi = \sqrt{\frac{m \omega}{\hbar}} \times \text{greek } \chi \]

\[ \frac{d^2 \psi}{d \xi^2} = (\xi^2 - K) \psi \]

\[ K = E / (\frac{1}{2} \hbar \omega) = 2E / \hbar \omega \]

\( \psi, K \) are dimensionless

Solve eq. \(*\) by power series

(i) General (ugly) way to solve d.e.

(ii) Pull out asympt. behavior first

(iii) Expand remainder in a power series

(iv) Impose b.c. to get energy eigenvalues c, to allowed values of c

(iii) For large \( \xi \)

\[ \frac{d^2 \psi}{d \xi^2} \lesssim \xi^2 \psi \]

\[ \psi = A e^{-\xi^2 / 2} \]

Bad \( n > 0 \) not normalized.

(iii) Guess \( \psi(\xi) = h(\xi)e^{-\xi^2 / 2} \)

For general \( h(\xi) \) this can still be exact.
\[
\frac{d\psi}{d\xi} = \left[ \frac{\partial h}{\partial \xi} - \xi \right] e^{-\xi^2/2}
\]
\[
\frac{\partial^2 \psi}{\partial \xi^2} = \left[ \frac{\partial^2 h}{\partial \xi^2} - 2\xi \frac{\partial h}{\partial \xi} + h \left( \xi^2 - 1 \right) \right] e^{-\xi^2/2}
\]

So
\[
\frac{\partial^2 \psi}{\partial \xi^2} = \left( m^2 - K \right) \psi
\]
\[
\left[ \frac{\partial^2 h}{\partial \xi^2} - 2\xi \frac{\partial h}{\partial \xi} + \left( K - 1 \right) h \right] = 0
\]

\[\xi^2 \text{ cancel}\]

Expand
\[
h = \sum_{j=0}^{\infty} a_j \xi^j
\]
\[
h' = \sum_{j=0}^{\infty} j a_j \xi^{j-1}
\]
\[
h'' = h'' = \sum_{j=0}^{\infty} j(j-1) a_j \xi^{j-2}
\]

Change dummy variables for \( h'' \)

\[
h'' = \sum_{i=0}^{\infty} (i+2)(i+1) a_{i+2} \xi^i, \quad i = j-2
\]
\[
\sum [ \sum_{i=0}^{\infty} (i+2)(i+1) a_{i+2} \xi^i - 2ia_i \xi^i + (K-1)a_i \xi^i ]
\]

If true for all \( \xi \) then each coeff. vanishes
\[(i+2)(i+1)a_{i+2} + K_{i-1} - \frac{i}{(i+2)(i+1)}a_i = 0\]

True for all \(i\):

\[a_{i+2} = \frac{2i+1-K}{(i+2)(i+1)}a_i\]

Reursion formula for \(\{a_i\}\) given \(a_i\):

Given \(a_0\) get \(a_2, a_4, \ldots \) a even.

Given arbitrary \(a_1\) get \(a_3, a_5, \ldots \) a odd.

Two arbitrary constants \(a_0, a_1\)

For 2nd side, dif. eq. these get all other \(a_i\).

For large \(i\):

\[a_{i+2} = \frac{2}{i} a_i\]

\[a_i = C \left(\frac{1}{i}\right)!\]

\[h(s) = C \sum_{i} \frac{s^i}{i!} = C e^{se^{s^2}}\]

\[\text{It will series does not terminate.}\]

\[\Psi(s) = h(s)e^{-s^{2}/2} \rightarrow e^{-s^{2}/2}\]

\[\text{want back asym. behavior we don't}\]
Solution to this problem is to choose some \( i = n \) such that
\[
2n+1 = K
\]
He\(n\) \( a_i \equiv 0 \) for \( i \geq n \)
because
\[
a_{i+2} = \frac{2i+1 - K}{(i+2)(i+1)} a_i
\]

\[
\psi = e^{i\phi/2} \quad K \neq 2n+1
\]

Energy eigenvalues
\[
K = 2n+1
\]
\[
E = \frac{1}{2} \hbar \omega \quad K = (n+\frac{1}{2}) \hbar \omega
\]
\( n = \) integer 0, 1, 2, ...
Ground state \( n = 0 \Rightarrow K = 1 \)

\[ a_2 = \frac{2 \cdot 0 + 1 - 1}{(2 + 0)(1 + 0)} \quad a_0 = 0 \]

\[ a_4 = \frac{3 - 1}{4 \cdot 3} \quad a_2 = \frac{2}{12} \quad a_2 = 0 = a_6 \ldots \]

\[ h(x) = a_0 \]

\[ \psi_0 = a_0 e^{-\frac{\xi^2}{2}} \]

\[ \xi = \left(\frac{m^2 \omega^2}{\hbar^2}\right)^{\frac{1}{2}} \]

\[ \psi_0(x) = a_0 e^{-\frac{mk^2}{\hbar} \frac{x^2}{2}} \quad E = \frac{\hbar \omega}{2} \]

Determine \( a_0 \) by normalization

\[ \int_{-\infty}^{\infty} \psi_0^* (x) \psi_0(x) \, dx = 1 \]

For \( n = 1 \) choose \( a_0 = c \)

\[ a_3 = \left[ (2n + 1) - k(n + 1) \right] a_0 \]

\[ a_1 = 0 = a_5 \ldots \]

\[ h(x) = a_1 \xi \]

\[ \psi_1 = a_1 \xi e^{-\frac{\xi^2}{2}} \quad E = \frac{3 \hbar \omega}{2} \]

\[ \psi_2 = a_0 (1 - 2\xi^2) e^{-\frac{\xi^2}{2}} \quad E = \frac{5 \hbar \omega}{2} \]