Lecture #5 Time-independent Schrodinger Equation

Last time $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

$$\hat{x} = x$$

Gaussian Wave Packet

$$\Psi = A e^{-\frac{\lambda}{2}x^2}$$

You calculate $\langle x \rangle = 0$ symmetry

$$\langle x^2 \rangle = 1/2\lambda$$

Use $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$

$$\langle p \rangle = \int dx\, A^* e^{-\frac{\lambda}{2}x^2} \left(-i\hbar \frac{\partial}{\partial x}\right) A e^{-\frac{\lambda}{2}x^2}$$

$$= A^* A\, i\hbar\lambda \int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda}{2}x^2}\, \underbrace{x\, e^{-\frac{\lambda}{2}x^2}}_{\text{odd}} \equiv 0$$

$$\hat{p}^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

$$\hat{p}^2 \Psi = -\hbar^2 \frac{\partial}{\partial x} A\left(-\lambda x\, e^{-\frac{\lambda}{2}x^2}\right)$$

$$= +\hbar^2 A \lambda \left[e^{-\frac{\lambda}{2}x^2} - \lambda x^2 e^{-\frac{\lambda}{2}x^2}\right]$$

$$= \hbar^2 \left[\lambda - \lambda^2 x^2\right] \Psi$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} dx\, \Psi^* \hat{p}^2 \Psi$$

$$= \hbar^2 \int_{-\infty}^{\infty} dx\, \Psi^* \left[\lambda - \lambda^2 x^2\right] \Psi$$

$$= \hbar^2 \left[\lambda - \lambda^2 \langle x^2 \rangle\right]$$

$$\langle p^2 \rangle = \hbar^2 \left[\lambda - \lambda^2 \frac{1}{2\lambda} \right] = \frac{\hbar^2 \lambda}{2}$$

$$\langle \Delta x^2 \rangle^{1/2} = \left[\langle x^2 \rangle - \langle x \rangle^2 \right]^{1/2} = \left[\frac{1}{2\lambda} - 0 \right]^{1/2}$$

$$\boxed{\Delta x = \frac{1}{(2\lambda)^{1/2}}}$$

$$\Delta p \equiv \langle (\Delta p)^2 \rangle^{1/2} = \left[\langle p^2 \rangle - \langle p \rangle^2 \right]^{1/2}$$

$$\boxed{\Delta p = \left[\hbar^2 \frac{\lambda}{2} - 0 \right]^{1/2} = \hbar \left(\frac{\lambda}{2} \right)^{1/2}}$$

and finally
$$\Delta p \, \Delta x = \hbar \left[\frac{\lambda}{2} \, \frac{1}{2\lambda} \right]^{1/2}$$

$$\boxed{\Delta p \, \Delta x = \frac{\hbar}{2}}$$

(a) Independent of λ

(b) You can trade: a large $\lambda \rightarrow$ small Δx but also must then have large Δp
Small $\lambda \longrightarrow$ Small Δp but large Δx

(c) Gaussian packet has minimum $\Delta p \, \Delta x$ other wave functions can have larger uncert.
Thus

Heisenberg Uncertainty Principle

$$\Delta p \, \Delta x \geq \frac{\hbar}{2}$$

(1) Can determine Δx <u>or</u> Δp to arbitrary accuracy. Just not both

(2) One way to think about it. Need at least one photon to bounce off an electron to see it.

Wavelength λ of photon restricts

$$\Delta x \sim \lambda$$

Need to use short wave length radiation to make good Δx measurement

But $p = h/\lambda = 2\pi\hbar/\lambda$ (deBroglie formula) is momentum of photon. In bouncing off the electron the recoil will impart a Δp to the electron

$$\Delta p \sim p \sim h/\lambda$$

Thus clearly Heisenberg U. principle is a limitation on our <u>information</u> about a particle.

However it goes much deeper, It is built into the formalism of quantum mechanics. You can not write down a wave function with better $\Delta p \, \Delta x$.

People go so far as to say extra information does not exist (Orthodox or Copenhagen interpretation). The particle did not have a position when I measured its momentum.

Chap. 2 Time-Independent Schrodinger equation

Want to solve Schrodinger equation.

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t)$$

Complicated partial dif. equation Ψ function both of x, t.

1st Thing to try because it is simple is seperation of variables. Can we write

$$\Psi(x,t) = \psi(x) f(t) \qquad \text{(A)}$$

[In general this will not be true. However even then we may be able to use a sum

$$\Psi(x,t) = \sum_i \psi_i(x) f_i(t)$$]

But if it is true it makes life much easier.

Plug (A) into S. eq.

$$\psi(x)\left[i\hbar \frac{\partial f(t)}{\partial t}\right] = f(t)\left(-\frac{\hbar^2}{2m}\right)\frac{\partial^2}{\partial x^2}\psi(x) + V(x) f(t)\psi(x)$$

Divide through by ψf

$$i\hbar \frac{1}{f}\frac{\partial f}{\partial t} = \frac{1}{\psi}\left[-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi(x)\right]$$

Left hand side only a function of t

Right hand side " " " " x,

if True for all t, x they must be equal to a constant.

Call the constant E (it turns out to be the particles energy but we will see that later)

$$i\hbar \frac{1}{f(t)} \frac{df}{dt} = E \qquad \text{B}$$

$$-\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{d^2\psi}{dx^2} + V = E$$

or $$\boxed{-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)} \qquad \text{C}$$

This is time-indep. Schrodinger equation

Can think of $\hat{p} = i\hbar \frac{\partial}{\partial x}$

$$\left[\frac{\hat{p}^2}{2m} + V(x)\right]\psi(x) = E\psi(x)$$

Compared to Classical mechanics

$$E = T + V = \frac{p^2}{2m} + V$$

Solve B

$$\int \frac{df}{f} = -\frac{i}{\hbar} E \int dt$$

$$\ln f = -\frac{iE}{\hbar} t + C$$

$$\Rightarrow \quad \boxed{f = e^{-i\frac{E}{\hbar}t}}$$

A If we can seperate variables then

$$\Psi(x,t) = \psi(x)\, e^{-i\frac{E}{\hbar}t}$$

is called a <u>stationary state</u> because prob. density $\Psi^*\Psi$ is time independent

$$\Psi^*\Psi(x,t) = \psi^*(x)\, e^{i\frac{E}{\hbar}t}\, \psi(x)\, e^{-i\frac{Et}{\hbar}}$$

$$= \psi^*(x)\,\psi(x) \quad \text{indep. of time}$$

(B) Stationary states have definite energy E

Operator corresponding to energy

$$\hat{H} = \frac{\hat{p}^2}{2m} + V$$

$$\langle \hat{H} \rangle = \langle E \rangle \quad \text{average energy}$$

Note $\quad \hat{H}\psi(x) = E\psi(x)$

So $\quad \int dx\, \psi^*(x)\, \hat{H}\, \psi(x) = \langle \hat{H} \rangle$

$$= \int dx\, \psi^*(x)\, E\, \psi(x) = E \int dx\, \psi^*(x)\psi(x) = E$$

Lets calculate spread in energy

$$\langle (\Delta E)^2 \rangle = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2$$

$$\langle \hat{H}^2 \rangle = \int dx\, \psi^*\, \hat{H}\, \hat{H}\psi$$

$$= \int dx\, \psi^*\, \hat{H}(E\psi) = E \int dx\, \psi^*\, \hat{H}\, \psi$$

$$= E^2$$

$$\langle (\Delta E)^2 \rangle = E^2 - E^2 = 0 \Rightarrow \Delta E = 0$$