Lecture #5 Time-independent

Schrödinger Equation

Last time: \( \hat{p} = -i \hbar \frac{\partial}{\partial x} \)
\( \hat{x} = x \)

Gaussian Wave Packet
\( \psi = A e^{-\frac{1}{2}x^2} \)

You calculate: \( \langle x \rangle = 0 \)
\( \langle x^2 \rangle = \frac{\hbar^2}{2} \lambda \) Symmetry

Use:
\( \int_{-\infty}^{\infty} dx \ e^{-\frac{1}{2}x^2} = \sqrt{\pi} \)

\( \langle p \rangle = \int_{-\infty}^{\infty} dx \ A^* \ e^{-\frac{1}{2}x^2} \ \hat{p} \ e^{-\frac{1}{2}x^2} \)
\( = A^* A \ \hbar \ \lambda \ \int_{-\infty}^{\infty} dx \ e^{-\frac{1}{2}x^2} \ \int_{-\infty}^{\infty} dx \ e^{-\frac{1}{2}x^2} = \frac{\hbar^2 \lambda}{4} \rightarrow \text{odd} \)

\( \hat{p}^2 = -\hbar^2 \frac{\partial^2}{\partial x^2} \)
\( \hat{p}^2 \psi = -\hbar^2 \ \frac{\partial^2}{\partial x^2} \ A(-\lambda x e^{-\frac{1}{2}x^2}) \)
\( = +\hbar^2 A \lambda \left[ e^{-\frac{1}{2}x^2} - \lambda^2 x^2 e^{-\frac{1}{2}x^2} \right] \)
\( = \hbar^2 \left[ \lambda - \lambda^2 x^2 \right] \psi \)

\( \langle p^2 \rangle = \int_{-\infty}^{\infty} dx \ \frac{\partial^2}{\partial x^2} \ \int_{-\infty}^{\infty} dx \ \psi^* \hat{p}^2 \psi \)
\( = \hbar^2 \int_{-\infty}^{\infty} dx \ \frac{\partial^2}{\partial x^2} \ \left[ \lambda - \lambda^2 x^2 \right] \psi \)
\( = \hbar^2 \left[ \lambda - \lambda^2 \langle x^2 \rangle \right] \)
\[ \langle p^2 \rangle = \frac{\hbar^2}{2} \left[ \frac{1}{\lambda} - \lambda \left( \frac{1}{2\lambda} \right) \right] = \frac{\hbar^2}{2} \]

\[ \langle \Delta x^2 \rangle^{\frac{1}{2}} = \left[ \langle x^2 \rangle - \langle x \rangle^2 \right]^{\frac{1}{2}} = \left[ \frac{1}{2\lambda} - 0 \right]^{\frac{1}{2}} \]

\[ \Delta x = \frac{1}{(2\lambda)^{\frac{1}{2}}} \]

\[ \Delta p = \left[ \langle p^2 \rangle - \langle p \rangle^2 \right]^{\frac{1}{2}} = \hbar \left( \frac{1}{2\lambda} \right)^{\frac{1}{2}} \]

\[ \Delta p \Delta x = \hbar \left[ \frac{1}{2\lambda} - \frac{1}{2\lambda} \right]^{\frac{1}{2}} \]

\[ \Delta p \Delta x = \frac{\hbar}{2} \]

(a) Independent of \( \lambda \)

(b) You can trade a large \( \lambda \) \( \Rightarrow \) small \( \Delta x \) but also must then have large \( \Delta p \). Small \( \lambda \) \( \Rightarrow \) small \( \Delta p \) but large \( \Delta x \)

(c) Gaussian packet has minimum \( \Delta p \Delta x \) other wave functions can have larger uncertainty.

Thus

Heisenberg Uncertainty Principle

\[ \Delta p \Delta x \geq \frac{\hbar}{2} \]

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1) Can determine $\Delta X$ or $\Delta p$ to arbitrary accuracy. Just not both.

2) One way to think about it:
Need at least one photon to bounce off an electron to see it.

Wavelength $\lambda$ of photon restricts

$$\Delta X \approx \lambda$$

Need to use short wavelength radiation to make good $\Delta X$ measurement.

But $p = h/\lambda = 2\pi n h/\lambda$

(The de Broglie formula)

is momentum of photon. In bouncing off the electron, the recoiling will impart a $\Delta p$

$$\Delta p \approx p \approx h/\lambda$$

Thus clearly Heisenberg's uncertainty principle is a limitation on our information about a particle.

However it goes much deeper: It is built into the formalism of quantum mechanics. You can not write down a wave function with better $\Delta p \Delta X$.

People go so far as to say extra information does not exist (orthodox Copenhagen interpretation). The particle did not have a position when I measured its momentum.
Want to solve Schrödinger equation.
\[
\frac{i\hbar}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x) \Psi(x,t)
\]
Complicated partial d.f. equation \( \Psi \) function both of \( x,t \).

1st thing to try because it is simple is separation of variables, Can we write
\[
\Psi(x,t) = \Phi(x) \Phi(t)
\]

In general this will not be true. However even then we may be able to use a sum
\[
\Psi(x,t) = \sum \Psi_i(x) f_i(t)
\]

But if it is true it makes life much easier.

Plug A into 5, eq.
\[
\Psi(x)[i\hbar \frac{df_i(t)}{dt}] = f_i(t) \left( -\frac{\hbar^2}{2m} \right) \frac{\partial^2 \Psi_i(x)}{\partial x^2} + V(x) \Phi_i(x)
\]

Divide through by \( \Psi f_i \)
\[
i\hbar \frac{1}{f_i} \frac{df_i(t)}{dt} = \frac{1}{\Psi} \left[ -\frac{\hbar^2}{2m} \frac{d^2 \Psi_i(x)}{dx^2} + V(x) \Psi_i(x) \right]
\]

Left hand side only a function of \( t \)
Right hand side only a function of \( x \) they must be equal to a constant.
Call the constant $E$ (it turns out to be the particle's energy but we will see that later)

\[ \hbar \frac{1}{f(x)} \frac{df}{dt} = E \]

\[ -\frac{\hbar^2}{2m} \frac{d^2 f}{dx^2} + V = E \]

or

\[ -\frac{\hbar^2}{2m} \frac{d^2 \Psi(x)}{dx^2} + V(x) \Psi(x) = E \Psi(x) \]

This is time-independent Schrödinger equation.

Can think of \( \hat{p} = i\hbar \frac{d}{dx} \)

\[ \left[ \frac{\hbar^2}{2m} + V(x) \right] \Psi(x) = E \Psi(x) \]

Compared to classical mechanics

\( E = T + V = \frac{p^2}{2m} + V \)

Solve \( \Box \)

\[ \int \frac{df}{f} = -i \hbar \int dt \]

\[ \ln f = -i \frac{E}{\hbar} t + C \]

\[ \Rightarrow \]

\[ f = e^{-i \frac{Et}{\hbar}} \]

\( \Box \) If we can separate variables then
\[ \psi(x,t) = \Psi(x) e^{-\frac{iEt}{\hbar}} \]

is called a stationary state because the probability density \( \psi^* \psi \) is time independent:

\[ \psi^* \psi(x,t) = \Psi^*(x) e^{\frac{Et}{\hbar}} \Psi(x) e^{-\frac{iEt}{\hbar}} = \Psi^*(x) \Psi(x) \quad \text{indep of time} \]

(6) Stationary states have definite energy operator, corresponding to energy:

\[ \hat{H} = \frac{\hat{p}^2}{2m} + V \]

\[ \langle \hat{H} \rangle = \langle E \rangle \quad \text{average energy} \]

Note: \[ \hat{H} \Psi(x) = E \Psi(x) \]

So:

\[ \int \Psi^*(x) \hat{H} \Psi(x) = \langle \hat{H} \rangle \]

\[ = \int \Psi^*(x) E \Psi(x) = E \int \Psi^* \Psi(x) \]

Let's calculate spread in energy:

\[ \langle (\Delta E)^2 \rangle = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2 \]

\[ \langle \hat{H}^2 \rangle = \int \Psi^* \hat{H}^2 \Psi = \int \Psi^* \hat{H} \hat{H} \Psi = \int \Psi^* \hat{H} (E \Psi) = E \int \Psi^* \Psi \hat{H} \Psi \]

\[ \langle (\Delta E)^2 \rangle = E^2 - E^2 = 0 \quad \Rightarrow \quad \Delta E = 0 \]