Lecture #4 Momentum and Uncertainty Principle

Last time

$\Psi^*(x,t)\,\Psi(x,t)$ = Probability density (prob. per unit length)

$\Psi^*(x)\Psi(x)\,dx$ = prob. to find particle between x and $x+dx$

Normalization

$$\int_{-\infty}^{\infty} dx\, \Psi^*(x,t)\,\Psi(x,t) = 1 \quad \text{(for all time?)}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\, \Psi^* \Psi = \int_{-\infty}^{\infty} dx\, \frac{\partial}{\partial t}\left[\Psi^*(x,t)\,\Psi(x,t)\right]$$

total derivative

partial derivative since also a function of x

$$\frac{\partial}{\partial t}\Psi^*\Psi = \Psi \frac{\partial}{\partial t}\Psi^* + \Psi^* \frac{\partial}{\partial t}\Psi$$

product rule

Use Schrodinger equation

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{iV}{\hbar}\Psi$$

$$\frac{\partial \Psi^*}{\partial t} = \frac{-i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{iV^*}{\hbar}\Psi^*$$

take complex conj $i \to -i$

assume V is real $V^* = V$

$$\frac{\partial}{\partial t}\Psi^*\Psi = \frac{i\hbar}{2m}\left[\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \Psi\frac{\partial^2\Psi^*}{\partial x^2}\right]$$

$$= \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right]$$

note $\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}$ terms cancel

$$\frac{d}{dt}\int dx\, \Psi^*\Psi = \frac{i\hbar}{2m}\int_{-\infty}^{\infty} dx\,\frac{\partial}{\partial x}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right]$$

$$= \frac{i\hbar}{2m}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right]\Bigg|_{-\infty}^{\infty}$$

But if $\int dx\, \Psi^*\Psi$ is to be finite at $t=0$
then

$$\boxed{\Psi \to 0 \quad \text{as} \quad x \to \pm\infty}$$

$\Rightarrow$ In general can drop surface terms as $x \to \pm\infty$

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx\, \Psi^*\Psi = 0$$

Thus

$$\boxed{\int_{\infty}^{\infty} dx\, \Psi^*(x,t)\,\Psi(x,t) = 1}$$

for all t

In QM time evolution is unitary
(keeps normalization unchanged)

Position: Mean value or expectation value

$$\langle x \rangle = \int_{-\infty}^{\infty} dx \; x \; \Psi^* \Psi$$

also $\langle x^2 \rangle = \int dx \; x^2 \; \Psi^* \psi$

$$\Delta x = \left[\langle x^2 \rangle - \langle x \rangle^2 \right]^{1/2}$$

Momentum

$$\langle p \rangle = m \langle v \rangle = m \frac{\partial}{\partial t} \langle x \rangle$$

Expect $v = \frac{\partial x}{\partial t}$ so $\langle v \rangle = \frac{\partial}{\partial t} \langle x \rangle$

$$\langle v \rangle = \frac{\partial \langle x \rangle}{\partial t} = \int dx \; x \; \frac{\partial}{\partial t} \left[\Psi^* \Psi \right]$$

$$= \frac{i\hbar}{2m} \int_{-\infty}^{\infty} dx \; x \; \frac{\partial}{\partial x} \left[\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right]$$

Integrate by parts

$$\int_a^b dU \; V = - \int_a^b dV \; U + UV \Big|_a^b$$

$\frac{\partial}{\partial x} x = 1$ and surface terms go away because $\Psi \to 0$ as $x \to \infty$

$$x \; \Psi^* \frac{\partial \Psi}{\partial x} \Big|_{-\infty}^{\infty} \equiv 0$$

$$\langle v \rangle = -\frac{i\hbar}{2m} \int_{-\infty}^{\infty} dx \left[\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right]$$

$$= -\frac{i\hbar}{m} \int_{-\infty}^{\infty} dx \; \Psi^* \frac{\partial \Psi}{\partial x}$$

integrate again by parts

$$\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} dx\, \Psi^* \frac{\partial}{\partial x} \Psi = m \langle v \rangle$$

Define an operator

$$\boxed{\begin{aligned} \hat{p} &\equiv -i\hbar \frac{\partial}{\partial x} \\ \langle p \rangle &= \int dx\, \Psi^* \hat{p}\, \Psi \end{aligned}}$$

In general for any operator $\hat{O}$

$$\langle \hat{O} \rangle = \int_{-\infty}^{\infty} dx\, \Psi^* \hat{O}\, \Psi$$

For $\langle x \rangle$ $\quad \hat{x} = x \quad$ multiplication by x

The Uncertainty principle

$$\langle (\Delta p)^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2$$

$$\langle \Delta p^2 \rangle^{1/2} \equiv \Delta p = [\langle p^2 \rangle - \langle p \rangle^2]^{1/2}$$

$$\langle p^2 \rangle = \int dx\, \Psi^* \hat{p}\, \hat{p}\, \Psi$$

$$= -\hbar^2 \int dx\, \Psi^* \frac{\partial^2}{\partial x^2} \Psi$$

Heisenberg Uncert. principle

$$\boxed{\Delta p \, \Delta x \geq \frac{\hbar}{2}}$$

For all states

Example

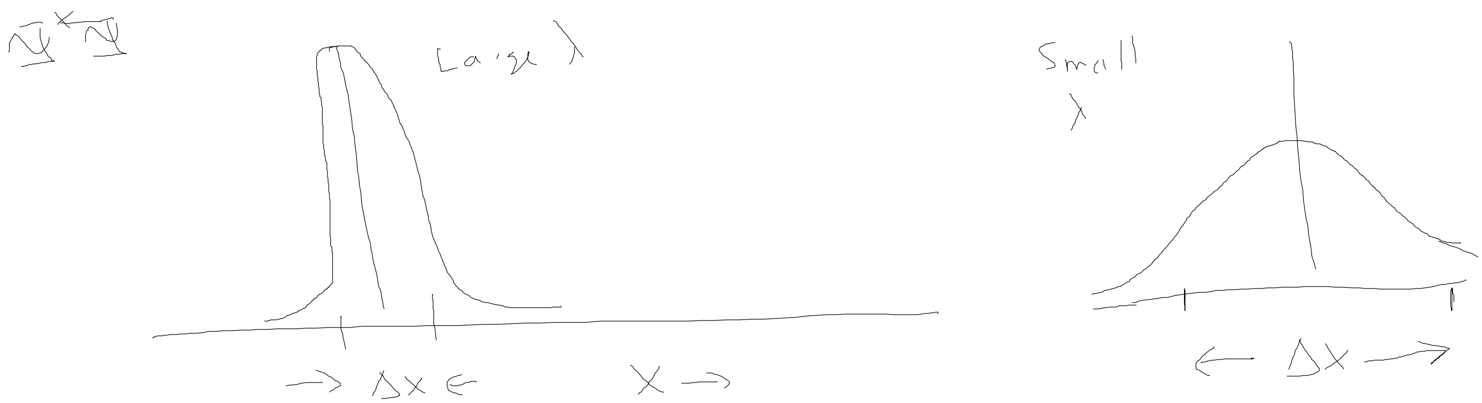
Gaussian Wave function

Important simple example. Gaussian wave function is smooth and provides minimum uncert.

$$\Psi(x,0) = A e^{-\frac{\lambda}{2}x^2}$$

You will show $\Delta x \sim \lambda^{-1/2}$

large λ sharply peaked wave function, small Δx



$$\langle p \rangle = A^* A \int dx\, e^{-\frac{\lambda}{2}x^2} \left(-i\hbar \frac{\partial}{\partial x}\right)\left(e^{-\frac{\lambda}{2}x^2}\right)$$

$$= A^* A \int_{-\infty}^{\infty} dx\, i\hbar\lambda x\, e^{-\lambda x^2}$$

But $\int_{-\infty}^{\infty} dx\, x\, e^{-\lambda x^2} = 0$ by symmetry thus

$$\boxed{\langle p \rangle = \langle x \rangle = 0}$$

$$\langle p^2 \rangle = A^* A \int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda}{2}x^2} \left(-i\hbar \frac{\partial}{\partial x}\right) \left[i\hbar\lambda\, x\, e^{-\frac{\lambda}{2}x^2}\right]$$

$$= \hbar^2 A^* A \int dx\, e^{-\lambda x^2}\left[\lambda - \lambda^2 x^2\right]$$

$$= \hbar^2\left[\lambda - \lambda^2\langle x^2\rangle\right]$$

Using normalization of wave function

You will show prob. 1.6

$$\langle x^2 \rangle = \frac{1}{2\lambda}$$

so $\boxed{\langle p^2 \rangle = \hbar^2 \frac{\lambda}{2}}$

Note need Gaussian integral

$$I = \int_{-\infty}^{\infty} dt \; e^{-t^2}$$

Can evaluate this with a trick

$$I^2 = \int_{-\infty}^{\infty} dt_x \int_{-\infty}^{\infty} dt_y \; e^{-(t_x^2 + t_y^2)}$$

Then go to cylindrical coordinates

$$r = (t_x^2 + t_y^2)^{1/2}$$

$$dt_x \, dt_y = r \, dr \, d\theta$$

and θ goes from $0 \to 2\pi$

$$\int_0^{2\pi} d\theta = 2\pi$$

$$\int dt_x \int dt_y \; e^{-(t_x^2 + t_y^2)} = \int_0^{\infty} r \, dr \; 2\pi \; e^{-r^2}$$

You show $\int_0^{\infty} r \, dr \, e^{-r^2} = \frac{1}{2}$ let $x = r^2$, $\frac{dx}{2} = r \, dr$

$$\Rightarrow I^2 = 2\pi \frac{1}{2} = \pi$$

$$\Rightarrow \boxed{I = \int_{-\infty}^{\infty} dt \; e^{-t^2} = \pi^{1/2}}$$

Can evaluate

$\int_{-\infty}^{\infty} dt \; t^2 e^{-t^2}$ by integration by parts.

With $\langle p \rangle = \langle x \rangle = 0$

$$\Delta p = \langle p^2 \rangle^{1/2} = \hbar \sqrt{\frac{\lambda}{2}}$$

$$\Delta x = \langle x^2 \rangle^{1/2} = \sqrt{\frac{1}{2\lambda}}$$

$$\boxed{\Delta p \, \Delta x = \frac{\hbar}{2}}$$ independent of λ!

(a) Well localized in space (small Δx) implies poor localization in momentum (large Δp)

(b) Gaussian wave function has minimum uncertainty; other kinds of wave functions can have $\Delta p \, \Delta x > \hbar/2$