Lecture #23 Final Review

Delta function pot.

\[ V = -\alpha \delta(x) \]

B.C. at position of delta function

1. Wavefunction is always cont.
2. Discont. in $d\psi/dx$ (only impact of delta function)

Integrate Schrodinger equation

\[-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi\]

Over small region from position of delta function $-\varepsilon \leq x \leq \varepsilon$. Take limit $\varepsilon \to 0$

\[-\frac{\hbar^2}{2m} \left( \frac{d\psi}{dx} \right)^2 - \frac{d\psi}{dx} \right] + \int_{-\varepsilon}^{\varepsilon} dx \ V(x) \psi(x) = E \int_{-\varepsilon}^{\varepsilon} dx \ \psi(x)\]

\[-\frac{\hbar^2}{2m} \left[ \frac{d\psi}{dx} \right]^2 - \frac{d\psi}{dx} \right] \bigg|_{-\varepsilon}^{\varepsilon} - \alpha \psi(0) = 0\]

\[E \approx \text{something} \]
Relation between Wave mechanics and matrix mechanics.

In Wave mechanics solve S. eq. in space

\[-\frac{h^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) = V(x) \psi(x) = E \psi(x)\]

Can always expand general wave function in an orthonormal basis

\[|\Psi\rangle = \sum_i c_i |e_i\rangle\]

Abstract state vector

Project on coordinate space basis

\[\psi(x) = \langle x | \Psi \rangle\]

\[e_i(x) = \langle x | e_i\rangle\]

\[\langle e_i | e_j \rangle = \delta_{ij} = \sum_k e_{ik}^* e_{jk}(x)\]

Start with

\[\hat{H} |\Psi\rangle = E |\Psi\rangle\]

Project out \(e_j\) matrix element (Take dot product with \(|e_j\rangle\)

\[\langle e_j | \hat{H} |\Psi\rangle = E \langle e_j | |\Psi\rangle\]

Use expansion for \(|\Psi\rangle\)
\[ \langle e_j | \hat{H} \Sigma c_i | e_i \rangle = E \langle e_j | c_i \rangle \]

Hamiltonian matrix

\[ H_{ji} = \langle e_j | \hat{\Sigma} | e_i \rangle = \int dx \, e_j^* (x) \hat{\Sigma} e_i (x) \]

State vector

\[ C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} \]

Schrödinger eq. is equiv. to matrix equation

\[ \begin{bmatrix} H \end{bmatrix} \begin{bmatrix} C \end{bmatrix} = E \begin{bmatrix} C \end{bmatrix} \]

Normalization

\[ \langle \psi | \psi \rangle = 1 \]

Wave mech.

\[ \int dx \, \overline{\psi} (x) \psi (x) = 1 \]

Matrix mech.

\[ C^T C = \Sigma c_i^* c_i = 1 \]
What did we use

(a) Completeness: can expand any function in a complete set.

(b) Orthogonality \( \langle e_i | e_j \rangle = 0 \quad i \neq j \)

(c) Normalization: The basis states are normalized \( \langle e_i | e_i \rangle = 1 \) so the expansion coefficients are also normalized \( \langle \psi | \psi \rangle = 1 \quad \Rightarrow \quad c^*c = 1 \)