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Lecture #21 Time dependence

Example: Problem 3.58

Important: this is QM!

$$H = \begin{bmatrix} h & g \\ g & h \end{bmatrix} \qquad \psi = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$a^* a + b^* b = 1$$

(a) Find eigenvalues and eigenvectors

$$H\psi = E\psi$$

$$\det \begin{bmatrix} h-E & g \\ g & h-E \end{bmatrix} = 0 \Rightarrow (h-E)^2 - g^2 = 0$$

$$E = h \pm g \qquad \boxed{\begin{matrix} E_1 = h-g \\ E_2 = h+g \end{matrix}}$$

$$\begin{bmatrix} h-E_1 & g \\ g & h-E_1 \end{bmatrix} \begin{bmatrix} \psi_1 \end{bmatrix} = 0$$

$$h - E_1 = g$$

$$\begin{bmatrix} g & g \\ g & g \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = 0 \Rightarrow a_1 = -b_1$$

$$\psi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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$h - E_2 = -g$

$$\begin{bmatrix} -g & g \\ g & -g \end{bmatrix} \begin{bmatrix} \psi_2 \end{bmatrix} = 0$$

$$\psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b) Given $\Psi(x,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Find $\Psi(x,t)$

(no x dependence)

Expand in eigenstates

$$\Psi(x,t) = c_1 \psi_1(x) e^{-iE_1 t/\hbar} + c_2 \psi_2 e^{-iE_2 t/\hbar}$$

Need c_1 and c_2

$$c_1 = \langle \psi_1 | \Psi \rangle = \frac{1}{\sqrt{2}} (1, -1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$c_2 = \langle \psi_2 | \Psi \rangle = \frac{1}{\sqrt{2}} (1, 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

used $\langle \psi_1 | \psi_1 \rangle = \langle \psi_2 | \psi_2 \rangle = 1$

$\langle \psi_1 | \psi_2 \rangle = 0$

$$\Psi(t) = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i(h-g)t/\hbar} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i(h+g)t/\hbar}$$

$$\Psi(t) = e^{-\frac{iht}{\hbar}} \begin{bmatrix} \frac{1}{2}(e^{igt/\hbar} + e^{-igt/\hbar}) \\ \frac{1}{2}(e^{-igt/\hbar} - e^{igt/\hbar}) \end{bmatrix}$$

$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

$$\boxed{\Psi(t) = e^{-\frac{iht}{\hbar}} \begin{bmatrix} \cos(gt/\hbar) \\ -i\sin(gt/\hbar) \end{bmatrix}}$$

System osc. in time at $t=0$
in state 1 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

at $gt/\hbar = \pi/2$ in state 2 $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example: Neutrino osc.
Neutrino spin $1/2$ low mass particle
needed for energy conservation in
nuclear decays.

Example: $n \to p + e^- + \nu_e$
In about 16 min.

ν_e is <u>Flavor</u> eigenstate, say $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
also ν_μ muon flavor neutrino
In $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$
makes pure flavor eigenstate $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

It is quite likely (if ν has a mass) that these flavor eigenstates are not energy eigenstates

$$H = \begin{bmatrix} h_1 & g_1 \\ g_1^* & h_2 \end{bmatrix} \quad \text{in} \quad \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$$

basis with $g_1 \neq 0$

Therefore a ν_e will have some nonzero prob. to osc. into a ν_μ as it travels from the sun to the Earth.

<u>Solar ν problem</u>: Several exp. see fewer ν_e from Sun than expected

<u>Atmospheric ν anom.</u>: Cosmic rays produce π which decay and produce ν_μ and ν_e in Earth's atmosphere. Deep underground detectors see fewer ν_μ than expected (relative to ν_e)

Could be $\nu_\mu \to \nu_\tau$ osc.

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Standard Model

quarks	u	c	t	top
	d	s	b	bottom
leptons	e^-	μ^-	τ^-	tau
	ν_e	ν_μ	ν_τ	tau neutrino
	1st generation	2nd	3rd generation	

Time dependence of expectation values

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\langle \Psi | Q \Psi\rangle$$
$$= \left\langle \frac{\partial \Psi}{\partial t} \middle| Q \Psi\right\rangle + \left\langle \Psi \middle| Q \frac{\partial \Psi}{\partial t}\right\rangle + \left\langle \Psi \middle| \frac{\partial Q}{\partial t} \Psi\right\rangle$$

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi\rangle$$

$$\frac{d}{dt}\langle Q\rangle = \frac{-1}{i\hbar}\langle H\Psi | Q\Psi\rangle + \frac{1}{i\hbar}\langle \Psi | QH\Psi\rangle + \left\langle \Psi \middle| \frac{\partial Q}{\partial t}\Psi\right\rangle$$

$$\langle \hat{H}\Psi | \hat{Q}\Psi\rangle = \langle \Psi | \hat{H}\hat{Q}\Psi\rangle \quad \text{note order}$$

$$\boxed{\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle [H, Q]\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle}$$

If $[H, Q] = 0$ then $\langle Q \rangle$ is a constant $\Longleftrightarrow$ Q is a conserved quantity   assumes $\frac{\partial Q}{\partial t} = 0$ (no explicit time dep.)

Consider

$$\sigma_H^2 \sigma_Q^2 \geq \left(\frac{1}{2i} \langle [H, Q] \rangle \right)^2$$

$$= \left(\frac{\hbar}{2}\right)^2 \left(\frac{d\langle Q \rangle}{dt} \right)^2$$

assume $\frac{\partial Q}{\partial t} = 0$

$$\sigma_H \sigma_Q \geq \frac{\hbar}{2} \left| \frac{d \langle Q \rangle}{dt} \right|$$

Define $\Delta t \equiv \sigma_Q / |d\langle Q \rangle / dt|$ length of time needed for Q to change by a significant fraction of σ_Q then (with $\Delta E = \sigma_H$)

$$\boxed{\Delta E \, \Delta t \geq \hbar/2}$$

Note Δt has no conventional definition in nonrel. quantum mec. Can measure position but not time of a particle. Time is just a parameter