Lecture #20 The Uncertainty Principle

General proof given two observables A, B with operators $\hat{A}$, $\hat{B}$ which do not commute $[\hat{A}, \hat{B}] \neq 0$ then

$$\sigma_A \sigma_B > 0$$

$$\sigma_A^2 = \langle A^2 \rangle - \langle A \rangle^2$$

$$= \langle (\hat{A} - \langle A \rangle)^2 \rangle$$

$$= \langle \underbrace{(\hat{A} - \langle A \rangle)}_{\text{this is Hermitian}} \Psi | (\hat{A} - \langle A \rangle) \Psi \rangle$$

$$= \langle f | f \rangle \qquad |f\rangle \equiv |(\hat{A} - \langle A \rangle) \Psi\rangle$$

$$\sigma_B^2 = \langle (\hat{B} - \langle B \rangle) \Psi | (\hat{B} - \langle B \rangle) \Psi \rangle$$

$$= \langle g | g \rangle \qquad |g\rangle \equiv |(\hat{B} - \langle B \rangle) \Psi\rangle$$

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$$

by Schwartz inequality

let $z = \langle f | g \rangle \qquad |z|^2 = (\mathrm{Re}\, z)^2 + \mathrm{Im}\, z^2$

$$|z|^2 \geq (\mathrm{Im}\, z)^2 = \left[\frac{1}{2i} (z - z^*) \right]^2$$

$$\mathrm{Im}\, z = \frac{1}{2i} (z - z^*)$$

$$\langle f | g \rangle = \langle (\hat{A} - \langle A \rangle) \Psi \,|\, (\hat{B} - \langle B \rangle) \Psi \rangle$$

$$= \langle \Psi | (\hat{A} - \langle A \rangle)(\hat{B} - \langle B \rangle) \Psi \rangle$$

$$= \langle \Psi | \hat{A} \hat{B} \Psi \rangle - \langle A \rangle \langle B \rangle$$

$$= \langle AB \rangle - \langle A \rangle \langle B \rangle$$

10/16

$$\langle g|f\rangle = \langle \hat{B}\hat{A}\rangle - \langle A\rangle\langle B\rangle$$

$$\langle f|g\rangle - \langle f|g\rangle^* = \langle \hat{A}\hat{B}\rangle - \langle \hat{B}\hat{A}\rangle$$
$$= \langle [\hat{A},\hat{B}]\rangle$$

$$\sigma_A^2\,\sigma_B^2 \geq \left[\mathrm{Im}\,\langle f|g\rangle\right]^2$$

$$\boxed{\sigma_A^2\,\sigma_B^2 \geq \left[\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right]^2}$$

Very general result

(a) If $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

$$[\hat{x},\hat{p}]\,f(x) = \hat{x}\,\frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}(x f)$$

$$= -\frac{\hbar}{i}f + x\frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}x\frac{\partial f}{\partial x}$$

so $$\boxed{[\hat{x},\hat{p}] = -\frac{\hbar}{i}}$$

$$\sigma_x^2\,\sigma_p^2 \geq \left(\frac{1}{2i}\,\cdot -\frac{\hbar}{i}\right)^2 = \left(\frac{\hbar}{2}\right)^2$$

so $$\boxed{\sigma_x\,\sigma_p \geq \frac{\hbar}{2}}$$

Any pair of observables whose operators do not commute can't be known with arbitrary precision. We call them <u>incompatible observables</u>.

Example: x, p

Two observables which do commute for example $x, V(x)$ can be known with arbitrary precision (but need not be)

If $[\hat{A}, \tilde{B}] = 0$ can have simultanious eigenfunctions

$$\hat{A} |\Psi_{ab}\rangle = a |\Psi_{ab}\rangle$$

at the same time

$$\hat{B} |\Psi_{ab}\rangle = b |\Psi_{ab}\rangle$$

Example $x, V(x)$

$$\hat{x}\, \delta(x - x') = x' \delta(\cancel{x} - x')$$

and

$$V(x)\, \delta(x - x') = V(x')\, \delta(x - x')$$

Thus the eigen value of $\hat{V}$ is $V(x')$

<u>Example:</u> x, H are in general incompatible

Problem 3.39

$$\hat{H} = \frac{\hat{p}^2}{2m} + V$$

$$[\hat{x}, \hat{H}] = \frac{1}{2m}[\hat{x}, \hat{p}^2] + \underbrace{[\hat{x}, \hat{V}(x)]}_{=0}$$

$$[\hat{x}, \hat{p}^2]f = -\hbar^2\left[x\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2}{\partial x^2}(xf)\right]$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}xf = \frac{\partial}{\partial x}(f + xf') = f' + f' + xf'' = 2f' + x\frac{\partial^2 f}{\partial x^2}$$

$$[\hat{x}, \hat{p}^2]f = -\hbar^2[\cancel{xf''} - 2f' - \cancel{xf''}]$$

$$= +2\hbar^2 f'$$

so $$[\hat{x}, \hat{p}^2] = 2\hbar^2\frac{\partial}{\partial x} = 2\hbar i\hat{p}$$

So

$$\sigma_x^2\sigma_H^2 = \left(\frac{1}{2i}\left\langle\left[\hat{x}, \frac{\hat{p}^2}{2m}\right]\right\rangle\right)^2$$

$$\left(\hbar\frac{\partial}{\partial x} = i\hat{p}\right)$$

$$= \left(\frac{\hbar}{2m}\langle\hat{p}\rangle\right)^2$$

$$\sigma_x\sigma_H = \frac{\hbar}{2m}|\langle p\rangle|$$

If stationary state $|\Psi\rangle$ real so $\langle p\rangle = 0$ and $\sigma_x = 0$ so only have $0=0$ ✓

Example: Diagonalize $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

(a) Eigenvalues $\begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) = 0$

$\lambda = 1$ or $\lambda = 2$

(b) Find eigenvectors ① $\lambda = 1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$a_1 + a_2 = a_1$$
$$2a_2 = a_2 \Rightarrow a_2 = 0$$

normalize $\tilde{a}^* a = (a_1^*, a_2^*) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = |a_1|^2 + |a_2|^2 = 1$

$a_1 = 1 \quad a_2 = 0 \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1^1 \\ a_2^1 \end{pmatrix}$

② $\lambda = 2$

$$a_1 + a_2 = 2a_1 \Rightarrow a_1 = a_2$$
$$2a_2 = 2a_2$$

$|a|^2 + |a_1|^2 = 1 \Rightarrow a_1 = \frac{1}{\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1^2 \\ a_2^2 \end{pmatrix}$

(c) $S^{-1}{}_{ij} = (a^j)_i = \begin{pmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}$

10/16

$$S^{-1}{}_{ij} = \begin{pmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}$$

$$S = +\sqrt{2} \begin{pmatrix} 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1 \end{pmatrix}^{T} = +\sqrt{2} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} +1 & -1 \\ 0 & +\sqrt{2} \end{pmatrix}$$

$$S\,S^{-1} = \begin{pmatrix} +1 & -1 \\ 0 & +\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$S\,M\,S^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \checkmark$$