# Lecture #19 Hilbert Space

Function spaces can be vector spaces
add two functions $\rightarrow$ get another function
multiply by scalar $\rightarrow$ " " "

Example $P(N)$ = Set of all $N-1$ order polynomials

Now go to infinite dim. spaces

$P(\infty)$ = set of all polynomials.

A special kind of infinite dim. function space is a Hilbert space

$\frac{\text{Hilbert space}}{\text{product space}}$ is a <u>complete</u> inner

Complete $\equiv$ includes all of its limits

Example

$$f_N(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^N}{N!}$$

is in $P(\infty)$ for any finite $N$ but limit $N \to \infty$
$= e^x$

is not in $P(\infty)$ since $e^x$ is not a polynomial.

For our Hilbert space we want all functions which have an inner product

$\Rightarrow$ All Square-integrable Functions.

$L_2(a,b)$ = square-integrable on $a$ to $b$

$L_2(-\infty, \infty) = L_2$ for short

Our wave functions live in Hilbert space $L_2$

The eigenfunctions of the Hermitian operators

$i\hat{D} = i\frac{\partial}{\partial x}$ and $\hat{X} = x$

are important.

$f_\lambda = A_\lambda e^{-i\lambda x}$ $\qquad g_\lambda = B_\lambda \, \delta(x-\lambda)$

The set of all eigenvalues $\lambda$ is <u>spectrum</u> of an operator.

$i\hat{D}$ and $\hat{x}$ have continuous spectra. Any value of $\lambda$ is allowed.

Note these eigenfunctions are not in our Hilbert space because they are not square-integrable

$$\int_{-\infty}^{\infty} dx \; f_\lambda^* f_\lambda = |A_\lambda|^2 \int_{-\infty}^{\infty} dx = \infty$$

$$\int dx \; g_\lambda^* g_\lambda = |B_\lambda|^2 \int_{-\infty}^{\infty} dx \; \delta(x-\lambda)\delta(x-\lambda) = \delta(\lambda-\lambda) = \infty$$

They do however have some kind of an orthogonality condition

$$\int dx \; f_\lambda^*(x) \, f_\mu(x) = A_\lambda^* A_\mu \int_{-\infty}^{\infty} e^{i\lambda x} e^{-i\mu x} dx$$

$$= |A_\lambda|^2 \, 2\pi \, \delta(\lambda-\mu)$$

$$\int dx \; g_\lambda(x)^* g_\mu(x) = |B_\lambda|^2 \, \delta(\lambda-\mu)$$

Chose normalization to get a simple delta function

$$A_\lambda = \frac{1}{\sqrt{2\pi}}$$

$$f_\lambda = \frac{1}{\sqrt{2\pi}} e^{-i\lambda x} \qquad \langle f_\lambda | f_\mu \rangle = \delta(\lambda - \mu)$$

This looks like discrete spectrum case for harmonic osc. for example

$$\langle \psi_n | \psi_m \rangle = \delta_{nm}$$

## Generalized Statistical Interpretation

(1) The state of a particle is represented by a normalized vector $|\Psi\rangle$ in the Hilbert space $L_2$

(2) Observable quantities, $Q(x, p, t)$ are represented by Hermitian operators $\hat{Q}(x, \frac{\hbar}{i}\frac{\partial}{\partial x}, t)$ the expectation value of $Q$ in $|\Psi\rangle$ is $\langle \hat{Q} \rangle = \langle \Psi | \hat{Q} \Psi \rangle$

(a) Expectation value: prepare many identical systems each in $|\Psi\rangle$ make one measurement on each system and average

(b) To get operator write classical function of $x$ and $p$ and replace $p \to \hat{p} \to \frac{\hbar}{i}\frac{\partial}{\partial x}$ and $x \to \hat{x} \to x$

Example $H = T + V = \frac{p^2}{2m} + V$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

3 A measurement of Q on a particle in the state $|\Psi\rangle$ is certain to return the value $\lambda$ iff $|\Psi\rangle$ is an eigenvector of $\hat{Q}$ with eigenvalue $\lambda$.

Example if measurement of energy is sure to give E then $\hat{H}|\Psi\rangle = E|\Psi\rangle$ $|\Psi\rangle$ is eigenstate of $\hat{H}$ with eigenvalue E.

3′ If you measure Q you are certain to get one of the eigenvalues of Q. The probability of getting $\lambda$ is the square of the $\lambda$ comp. of $|\Psi\rangle$

Example measure E for Harmonic osc. can only get $(n+\frac{1}{2})\hbar\omega$ no other allowed result. Prob of $(N+\frac{1}{2})\hbar\omega$ is $|c_N|^2$: $|\Psi\rangle = \sum_i c_i|\psi_i\rangle$

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Two cases (A) spectrum <u>descrete</u>

$$\hat{Q}\,|e_n\rangle = \lambda_n\,|e_n\rangle \qquad n=1,2,3\ldots$$

$$\langle e_n|e_m\rangle = \delta_{nm}$$

$|e_n\rangle$ forms a basis. Completeness says can expand and state

$$|\Psi\rangle = \sum_{n=1}^{\infty} c_n\,|e_n\rangle$$

$$c_n = \langle e_n|\Psi\rangle$$

$$P_n = |c_n|^2 = |\langle e_n|\Psi\rangle|^2$$

(B) Spectrum <u>continuous</u>

$$\hat{Q}\,|e_k\rangle = \lambda_k\,|e_k\rangle \qquad -\infty < k < \infty$$

$$\langle e_k|e_\ell\rangle = \delta(k-\ell)$$

$$|\Psi\rangle = \int_{-\infty}^{\infty} dk\; c_k\,|e_k\rangle \qquad \text{(completeness of eigenstates)}$$

$$c_k = \langle e_k|\Psi\rangle$$

Prob. of getting measurement in range $dk$ about $\lambda_k$ is $P_k = |c_k|^2\,dk$
$= |\langle e_k|\Psi\rangle|^2\,dk$

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Our old coordinate space wave function is just a special case. $|\Psi\rangle$ lives in abstract Hilbert space

Position eigenstates $e_{x'}(x) = \delta(x-x')$

$$\Psi(x',t) = \langle e_{x'}|\Psi\rangle = \int_{-\infty}^{\infty} dx\, \delta(x-x')\,\Psi(x,t)$$
$$= \Psi(x',t)$$

To get momentum space wave function project $|\Psi\rangle$ onto momentum eigenstates

$$e_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar}$$

$$c_p = \langle e_p|\Psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \Psi(x,t)\, dx \equiv \Phi(p,t)$$

$\Phi$ is momentum-space wave function
Prob to get momentum between $p$ and $p+dp$ is
$$|\Phi(p,t)|^2 \, dp$$

Can project $|\Psi\rangle$ onto any basis
Note $x$ and $p$ eigenstates (basis functions) don't live in $L_2$!