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Lecture #18 Function Spaces.

Functions as vectors:
Sum of two functions is another function
multiply function by scalar → get another function

<u>Inner product</u> of two functions

$$\langle f | g \rangle \equiv \int f(x)^* \, g(x) \, dx$$

(limits depend on functions in question)

It is necessary that every admissible function be <u>square</u> <u>integrable</u>

$$\int f(x)^* \, f(x) \, dx < \infty$$

otherwise inner prod. does not exist. In fact this is sufficient

<u>Example</u> Consider set $P(N)$ of all polynomials of degree $< N$

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{N-1} x^{N-1}$$

on interval $-1 \le x \le 1$. They are square integrable

One basis (powers of x)

$$|e_1\rangle = 1 \quad |e_2\rangle = x \quad |e_3\rangle = x^2 \quad |e_N\rangle = x^{N-1}$$

This is not an orthonormal basis

$$\langle e_1 | e_3 \rangle = \int_{-1}^{1} dx \, x^2 = 2/3$$

Use Gram-Schmidt to get Legendre polynomials

$$|e_n'\rangle = \sqrt{n - 1/2} \; P_{n-1}(x)$$

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Example

$$|e_1'\rangle = |e_1\rangle / \|e_1\| = 1/\sqrt{2}$$

$$\|e_1\| = \left[\int_{-1}^{1} dx \; 1\right]^{1/2} = \sqrt{2}$$

$$|e_2'\rangle = \{|e_2\rangle - \langle e_1'|e_2\rangle |e_1'\rangle\} / \|e_2'\|$$

but $\langle e_1'|e_2\rangle = \int_{-1}^{1} dx \frac{1}{\sqrt{2}} x = 0$

$$\langle e_2|e_2\rangle = \int_{-1}^{1} dx \, x^2 = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3}$$

$$|e_2'\rangle = \sqrt{\frac{3}{2}}\, x = \sqrt{2 - \tfrac{1}{2}}\; P_{n-1} = \sqrt{2 - \tfrac{1}{2}}\; x$$

Operators as linear transformations

$$\hat{T} f(x) = \lambda f(x)$$

Eigenvectors of $\hat{T}$ are called <u>eigenfunctions</u>

Example $\hat{D} = \frac{d}{dx}$

$$\hat{D} f = \lambda f \qquad f = Ae^{\lambda x}$$

In $P(N)$ only eigenfunction corresponds to $\lambda = 0$

Hermitian operator

$$\langle f|\hat{T} g\rangle = \langle \hat{T} f|g\rangle$$

For all f, g in space. Is $\hat{D}$ hermitian?

$$\langle f|\hat{D} g\rangle = \int_a^b f^* \frac{dg}{dx} dx = \left. f^* g\right|_a^b - \langle \hat{D} f|g\rangle$$

Can get rid of minus sign
$\hat{D} = \frac{d}{dx}$ is not Hermitian but
$i\frac{d}{dx}$ may be. Minus sign canceled
by $i^* = -i$.

To get rid of boundary term: require functions in space to satisfy

$$f(a) = f(b)$$

for all f. In practice on $(-\infty, \infty)$ and square integrability implies $f(-\infty) = 0 = f(\infty)$

Now consider spaces of <u>infinite dimension</u>

The operator $\hat{x}$ is not a linear transformation $P(N)$ since it generates polynomials of order ~~dimension~~ $N+1$ which are only in $P(N+1)$

However $\hat{x}$ is a linear transformation on $P(\infty)$ space of all polynomials on $-1 \le x \le 1$
In fact its Hermitian

$$\int_{-1}^{1} f^*(x) \, [x \, g(x)] \, dx = \int_{-1}^{1} [x f(x)]^* \, g(x) \, dx$$

But what are its eigenfunctions? They are not polynomials

$$g_\lambda(x) = B \, \delta(x - \lambda)$$

$$\hat{x} \, g_\lambda(x) = \lambda \, g_\lambda(x)$$

Operator $\hat{x}$ has no eigenfunctions in $P(\infty)$

Last time I said
(a) Eigenvalues of Hermitian op. real
(b) Eigenfunctions for different eigenvalues orthogonal
(c) Eigenvectors of Hermitian trans. span space.

(c) Is only true in general for finite dim spaces. Will come back to this since (c) is so important for QM.

Hilbert Space

Subtle point: We wish to complete our function space. Example $P(\infty)$ includes

$$f_N(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \frac{x^N}{N!}$$

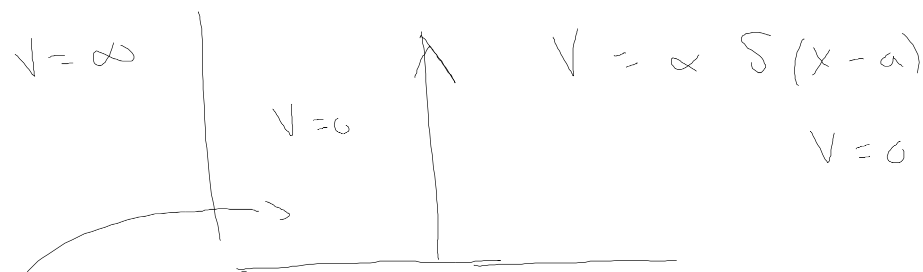
For any N but it does not include limit as $N \to \infty$

$$\lim_{N\to\infty} f_N(x) = \sum_{n=0}^{\infty} \frac{x^n}{N!} = e^x$$

A complete inner product space is called a Hilbert space. Complete means includes all its limits.

If we include limit $N \to \infty$ of all polynomials we get set of all square integrable functions on $-1, 1$ $L_2(-1, +1)$. We will ~~be~~ deal mostly with the set of all square integrable functions on $-\infty, \infty$ $L_2(-\infty, \infty)$ or L_2 for short. This is where wave functions live.

Example Problem 2.46



$$\psi = A\left(e^{ikx} - e^{-ikx}\right) \Leftarrow \psi(0) = 0$$

Outside $x > a$ $\quad \psi = B e^{ikx} + \cancel{C e^{-ikx}} = 0$

(a) $$\boxed{A\left(e^{ika} - e^{-ika}\right) = B e^{ika}}$$ ψ is cont.

$$\Delta \frac{\partial \psi}{dx} = \frac{2m\alpha}{\hbar^2}\psi(a)$$

$$\frac{1}{\psi}\frac{\partial \psi}{dx}\bigg|_{a+\epsilon} - \frac{1}{\psi}\frac{\partial \psi}{dx}\bigg|_{a-\epsilon} = \frac{2m\alpha}{\hbar^2}$$

From $$\int_{a-\epsilon}^{a+\epsilon}\left[-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{dx^2} + V\psi\right]dx = E\int_{a-\epsilon}^{a+\epsilon}\psi\, dx = 0$$

given $\hat{H}\psi = E\psi$

$$\frac{ikBe^{ikx}}{Be^{ikx}} = \frac{1}{\psi}\frac{\partial \psi}{dx}\bigg|_{a+\epsilon} = ik$$

$$ik\frac{\left(e^{ika} + e^{-ika}\right)}{e^{ika} - e^{-ika}} = \frac{1}{\psi}\frac{\partial \psi}{dx}\bigg|_{a-\epsilon}$$

$$ik - ik\left(\frac{e^{ika} + e^{-ika}}{e^{ika} - e^{-ika}}\right) = \frac{2m\alpha}{\hbar^2}$$

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$$\frac{-2ik\, e^{-ika}}{e^{ika} - e^{-ika}} = \frac{2m\alpha}{\hbar^2}$$

$$1 = \frac{m\alpha}{-ik\hbar^2}\left(e^{2ika} - 1\right)$$

$$\boxed{1 = i\frac{m\alpha}{k\hbar^2}\left(e^{2ika} - 1\right)}$$

Trans. eq. gives complex k

$$E = \frac{\hbar^2 k^2}{2m} = E_0 + i\Gamma \qquad \Gamma \neq 0$$

$$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$$

$$\Psi^*(x,t)\,\Psi(x,t) = \psi^*(x)\psi(x)\, e^{i(E^*-E)t/\hbar}$$

$$= \psi^*(x)\psi(x)\, e^{+2\,\mathrm{Im}E\, t/\hbar} = \psi^*\psi(x) e^{\mp 2\Gamma t/\hbar}$$

$$E^* - E = -2i\,\mathrm{Im}E = -2i\Gamma$$

Prob. to decay to $1/e$ is $t_{1/e} = -\hbar/2\Gamma$
mean life of resonance