10/9/98

Lecture #17 Matricies

Example: Pauli Spin Matricies 2×2
Hermitian matricies

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Properties of σ_i $\qquad i = x, y, z$

$\sigma_i^+ = \sigma_i$ Hermitian

Example $\quad \sigma_y^+ = \tilde{\sigma}_y^* = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^* = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_i^2 = I \qquad \sigma_y \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$\left[\underline{\underline{A}}\ \underline{\underline{B}}\right]_{ik} = \sum_k A_{ij} B_{jk}$$

$$\sigma_y \sigma_y = \begin{bmatrix} 0 \cdot 0 + -i \cdot i, & 0 \cdot (-i) + -i \cdot 0 \\ i \cdot 0 + 0 i, & i \cdot (-i) + 0 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Commutator

$$[\sigma_z, \sigma_x] \equiv \sigma_z \sigma_x - \sigma_x \sigma_z$$

$$\sigma_z \sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\sigma_x \sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$[\sigma_z, \sigma_x] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = +2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= 2i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2i\,\sigma_y$$

$\boxed{\text{Eigenvalues and Eigenvectors}}$

Given a transformation $\hat{T}$ there are special vectors $|\alpha\rangle$ such that

$$\hat{T}|\alpha\rangle = \lambda|\alpha\rangle$$

Gives back a multiple λ of the original vector. λ is called the eigenvalue and $|\alpha\rangle$ an eigen vector.

Note see later Schrodinger eq. has this form

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

with E the eigenvalue

Given a basis we have a Matrix eq.

$$[\underline{\underline{T}}]\,[a] = \lambda\,[a]$$

or

$$[\underline{\underline{T}} - \lambda\,\underline{\underline{I}}]\,[a] = [\underline{\underline{0}}]$$

$n \times n$ matrix of zeros

identity $n \times n$ matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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If $\left[\underline{\underline{T}} - \lambda \underline{\underline{I}}\right]^{-1}$ exists then

$$[T - \lambda I]^{-1} [T - \lambda I] [a] = 0$$

$$\Rightarrow \quad [a] = 0$$

So to have a nonzero eigenvector must have $[T - \lambda I]$ be a singular matrix with det. $= 0$.

$$\det \left[\underline{\underline{T}} - \lambda \underline{\underline{I}}\right] = 0$$

This is an equation for the allowed eigenvalues λ

Example: Find eigenvalues of

$$M = \begin{bmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{bmatrix}$$

$$|M - \lambda I| = \begin{vmatrix} (2-\lambda) & 0 & -2 \\ -2i & (i-\lambda) & 2i \\ 1 & 0 & (-1-\lambda) \end{vmatrix}$$

$$= (2-\lambda) \quad (i-\lambda)(-1-\lambda) - 0$$

$$- 2(\cancel{-1})(-\lambda + i)$$

$$= -(2-\lambda)(i-\lambda)(1+\lambda) + 2(i+\lambda)$$

$$= -\lambda^3 + (1+i)\lambda^2 - i\lambda = 0$$

Det is a Polynomial of order n

Roots of Polynomial are 0, 1 and i

$n \times n$ matrix has at least one and at most n distinct eigenvalues.

To find eigenvector for $\lambda = 1$ say

$$[M - I][a] = 0$$

$$\begin{bmatrix} 1 & 0 & -2 \\ -2i & i-1 & 2i \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

$$a_1 - 2a_3 = 0$$
$$-2i\,a_1 + (i-1)\,a_2 + 2i\,a_3 = 0$$
$$a_1 - 2a_3 = 0$$

redundant (singular matrix)

let $a_1 = 1$ say $\Rightarrow a_3 = +\frac{1}{2}$

$$-2i + i + (i-1)\,a_2 = 0$$

$$a_2 = i/(i-1) = \frac{i}{(i-1)} \frac{1+i}{(i+1)} = \frac{i-1}{-1-1}$$

$$a_2 = \frac{1}{2} - \frac{i}{2} = \frac{1}{2}(1-i)$$

$$\underline{\underline{a}}_1 = \begin{bmatrix} 1 \\ \frac{1}{2}(1-i) \\ \frac{1}{2} \end{bmatrix} \quad \text{for } \lambda = 1$$

label eigenvector by its eigenvalue

Hermitian Transformations

(a) $T^{+} = \tilde{T}^{*}$

(b) $\langle T^{+}\alpha | \beta \rangle = \langle \alpha | \hat{T} \beta \rangle$

These are equivalent

$$\langle \alpha | \beta \rangle = a^\dagger b$$

so $\langle \alpha | T \beta \rangle = a^\dagger T b = (T^\dagger a)^\dagger b$

$[a^\dagger T = (T^\dagger a)^\dagger]$ $\quad = \langle T^\dagger \alpha | \beta \rangle$

If $T^\dagger = T$ then

(1) Eigenvalues of Hermitian transformation are real

$$\hat{T} |\alpha\rangle = \lambda |\alpha\rangle \qquad \text{with } |\alpha\rangle \neq 0$$

calculate

$$\langle \alpha | T \alpha \rangle = \lambda \langle \alpha | \alpha \rangle$$

$$= \langle T \alpha | \alpha \rangle = \langle \lambda \alpha | \alpha \rangle$$

$$= \lambda^* \langle \alpha | \alpha \rangle$$

remember $\langle \alpha | \beta \rangle = \sum_i a_i^* b_i$ so if multiply each a_i by λ changes dot product by λ^*

Thus $\lambda \langle \alpha | \alpha \rangle = \lambda^* \langle \alpha | \alpha \rangle$ but $|\alpha\rangle \neq 0$

$\Rightarrow \boxed{\lambda = \lambda^*}$ eigenvalues real.

(2) Eigenvectors (of Herm. trans.) for dist. eigenvalues are orthogonal

$$T|\alpha\rangle = \lambda |\alpha\rangle \qquad T|\beta\rangle = \mu |\beta\rangle$$

$$\langle \alpha | T \beta \rangle = \mu \langle \alpha | \beta \rangle$$

$$= \langle T \alpha | \beta \rangle = \lambda \langle \alpha \beta \rangle$$

Thus $\boxed{\langle \alpha | \beta \rangle = 0}$

(3) Eigenvectors of a Hermitian trans. <u>span</u> the space

If n distinct eigenvalues $\Rightarrow$ n linearly indep eigenvectors which clearly span n dim. space

If eigenvalue λ is m fold degenerate have m possibly non orthogonal eigenvectors with same eigenvalue. However can orthogonalize these using Gram - Schmidt

Note these results will be seen to imply the following features of the Harmonic osc. solutions

(1) Energy E is real $\hat{H}|\psi\rangle = E\psi$

$$E = (n+\tfrac{1}{2})\hbar\omega$$

(2) $$\int_{-\infty}^{\infty} dx\, \psi_n^*(x)\, \psi_m(x) = 0 \qquad n \neq m$$

(3) Can expand any wave function

$$\Psi(x) = \sum_n c_n \psi_n(x)$$

For some set c_n. Thus the eigenfunctions ψ_n form a complete set.