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Lecture #16

Last time: inner product is complex #

$$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$$

Norm (length) $\|\alpha\| = \sqrt{\langle\alpha|\alpha\rangle}$

A set of vectors is <u>orthonormal</u> if $\langle\alpha_i|\alpha_j\rangle = \delta_{ij}$

W.r.to an orthonormal basis

$$\langle\alpha|\beta\rangle = \sum_{i=1}^{n} a_i^* b_i$$

$$\langle\alpha|\alpha\rangle = \sum_i |a_i|^2$$

The components are $a_i = \langle e_i|\alpha\rangle$ with $|e_i\rangle$ the basis vectors.

Schwarz inequality

Generalization of angle between two vectors with $|\cos\theta| \le 1$

$$|\langle\alpha|\beta\rangle|^2 \le \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$$

<u>Linear transformations</u> (related to operators such as $\hat{x} = x$ and $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

$\hat{T}$ transforms each vector $|\alpha\rangle \to |\alpha'\rangle = \hat{T}|\alpha\rangle$

<u>linear</u> $\hat{T}(a|\alpha\rangle + b|\beta\rangle) = a\hat{T}|\alpha\rangle + b\hat{T}|\beta\rangle$

If you know what $\hat{T}$ does to each basis vector

$$\hat{T}|e_1\rangle = T_{11}|e_1\rangle + T_{21}|e_2\rangle + \dots T_{n1}|e_n\rangle$$

$$\hat{T}|e_j\rangle = \sum_i \hat{T}_{ij}|e_i\rangle$$

for arbitrary $|\alpha\rangle = \sum_i a_i |e_i\rangle$

$$\hat{T}|\alpha\rangle = \sum_j a_j \hat{T}|e_j\rangle = \sum_j \sum_i a_j T_{ij}|e_i\rangle$$

$$= \sum_i \sum_j T_{ij}\, a_j\, |e_i\rangle$$

$\hat{T}$ takes a vector with components a_j and gives new vector $a'_i = \sum_j T_{ij} a_j$

This is matrix multiplication. Think of $\hat{T}$ as a matrix

$$\hat{T} = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$$

$n \times n$ square matrix

$n \times 1$ column vector $n \times n$ matrix $n \times 1$ column vector

$$\begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix} = \begin{pmatrix} \\ \hat{T} \\ \\ \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Sum of two linear transformations

$$(\hat{S} + \hat{T})|\alpha\rangle = \hat{S}|\alpha\rangle + \hat{T}|\alpha\rangle$$

$$\underline{\underline{U}} = \underline{\underline{S}} + \underline{\underline{T}} \qquad U_{ij} = T_{ij} + S_{ij}$$

Product of two transformation: $\hat{S}\hat{T}$
first do $\hat{T}$ then do $\hat{S}$ on resulting vector

$$|\alpha\rangle \to |\alpha'\rangle = \hat{T}|\alpha\rangle \to |\alpha''\rangle = \hat{S}|\alpha'\rangle$$

$$|\alpha''\rangle = \hat{S}(\hat{T}|\alpha\rangle)$$

$$a_i'' = \sum_j S_{ij} \sum_k T_{jk} a_k = \sum_k U_{ik} a_k$$

Matrix multiplication

$$\underline{\underline{U}} = \underline{\underline{S}}\,\underline{\underline{T}} \qquad \boxed{U_{ik} = \sum_j S_{ij} T_{jk}}$$

Transpose (tilda)

$$\tilde{T} = \begin{pmatrix} T_{11} & T_{21} & T_{n1} \\ T_{12} & T_{22} & T_{n2} \\ \vdots & & \\ T_{1n} & T_{2n} & T_{nn} \end{pmatrix}$$

Interchage i and j $\quad \tilde{T}_{ij} = T_{ji}$

$\tilde{a}$ = a $1 \times n$ row matrix

$$\tilde{a} = (a_1 \quad a_2 \quad \cdots \quad a_n)$$

a matrix is <u>symmetric</u> if $\tilde{T} = T$

antisym. if $\tilde{T} = -T$

Complex conjugate of a matrix is complex conjugate of each element

$$\underline{\underline{T}}^* = \begin{pmatrix} T_{11}^* & T_{12}^* & \cdots & T_{1n}^* \\ \vdots & & & \\ & & & T_{nn}^* \\ T_{n1}^* & & & \end{pmatrix}$$

<u>Hermitian conjugate</u> or <u>adjoint</u> of a matrix is transposed conjugate

$$\underline{\underline{T}}^\dagger = \tilde{T}^* = \begin{pmatrix} T_{11}^* & T_{21}^* & & T_{n1}^* \\ & & & \\ T_{1n}^* & & & T_{nn}^* \end{pmatrix}$$

$$a^\dagger = (a_1^*, a_2^* \ldots a_n^*)$$

Matrix is <u>Hermitian</u> if $T^\dagger = T$
Skew Hermitian if $T^\dagger = -T$

Inner product of two vectors

$$\langle \alpha | \beta \rangle = \underline{\underline{a}}^\dagger \underline{\underline{b}}$$

Careful of order $\underline{\underline{S}}\,\underline{\underline{T}} \neq \underline{\underline{T}}\,\underline{\underline{S}}$

<u>Commutator</u> $[\underline{\underline{S}}, \underline{\underline{T}}] = \underline{\underline{S}}\,\underline{\underline{T}} - \underline{\underline{T}}\,\underline{\underline{S}}$

$$(ST)^\dagger = T^\dagger S^\dagger \neq S^\dagger T^\dagger$$

Unit matrix $I_{ij} = \delta_{ij}$ $\underline{\underline{I}} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$

<u>Inverse</u>

$$\underline{\underline{T}}^{-1}\,\underline{\underline{T}} = \underline{\underline{T}}\,\underline{\underline{T}}^{-1} = \underline{\underline{I}}$$

Matrix has an inverse iff <u>determinant</u> is nonzero

$$(\underline{\underline{S}}\,\underline{\underline{T}})^{-1} = \underline{\underline{T}}^{-1}\,\underline{\underline{S}}^{-1}$$

order switched

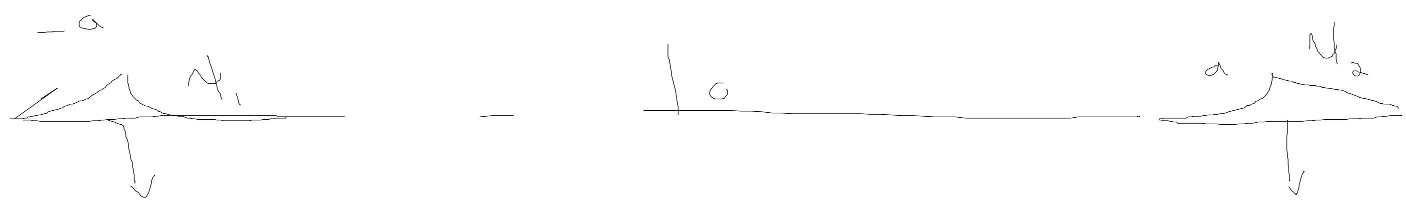
A matrix is <u>unitary</u> if

$$\underline{\underline{U}}^{\dagger} = \underline{\underline{U}}^{-1}$$

$\boxed{\text{Example Problem 2.26}}$ $V = -\alpha\,[\delta(x+a) + \delta(x-a)]$

(a) Sketch V

(b) How many bound states does it have?
Think of two isolated δ functions in limit $a \to \infty$



Pot. is symmetric $x \to -x$ so can find solutions <u>even</u>

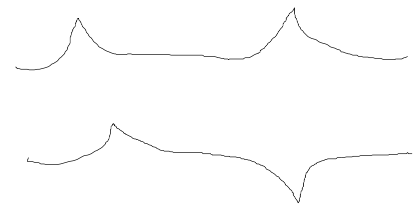
$$\psi(x) = \psi(-x) \qquad \psi \sim \psi_1(x) + \psi_2(x)$$

or <u>odd</u> $\psi(x) = -\psi(-x)$

$$\psi \sim \psi_1(x) - \psi_2(x)$$

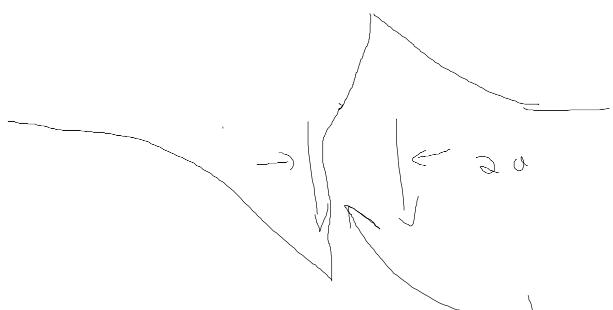
Thus expect <u>at most</u> <u>two</u> bound states. Since each isolated δ func. has only one bound state.

Expect even state to have lower E than odd state because odd state has an extra node.



If delta functs are very close together ~~even~~ it looks like one delta function with strength 2α. In this limit only one bound state.

odd solution as $a \to 0$



Thus even solution always exists. Odd solution may not exist if a is too small.

Look for even solution $E < 0$

$$\psi = A\left[e^{\kappa x} + e^{-\kappa x}\right] \qquad -a < x < a$$

$$\psi = B e^{-\kappa x} \qquad x > a$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\Rightarrow \quad \kappa = \frac{\sqrt{-2mE}}{\hbar} \qquad \text{for } E < 0$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

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Integrate S. eq.

$$-\frac{\hbar^2}{2m}\left[\left.\frac{d\psi}{dx}\right|_{a+\epsilon} - \left.\frac{d\psi}{dx}\right|_{a-\epsilon}\right] - \alpha\,\psi(a) = 0$$

$$-\frac{\hbar^2}{2m}\left[-\kappa\psi - \kappa A\left[e^{\kappa a} - e^{-\kappa a}\right]\right] = \alpha\psi$$

$$-\kappa - \kappa\frac{e^{\kappa a} - e^{-\kappa a}}{e^{\kappa a} + e^{-\kappa a}} = -\frac{2m\alpha}{\hbar^2}$$

$$2\kappa\left[\frac{e^{\mp\kappa x}}{e^{\kappa x} + e^{-\kappa x}}\right] = \frac{2m\alpha}{\hbar^2}$$

$$\boxed{\kappa\left[1 + e^{-2\kappa a}\right]^{-1} = \frac{m\alpha}{\hbar^2}}$$

even solution

Can solve numerical for κ

Given $\kappa \to E = -\frac{\hbar^2}{2m}\kappa^2$

Note this equation always has a solution.

For odd solution $\psi = \begin{cases} A\left[e^{\kappa x} - e^{-\kappa x}\right] \\ B e^{-\kappa x} & x > a \end{cases}$

as before except now

$$\kappa\left[1 - e^{-2\kappa a}\right]^{-1} = \frac{m\alpha}{\hbar^2}$$

This does not have a solution if a is to small. I.E. left hand side always greater then r. hand side