Lecture #16

Last time: inner product is complex #
\[ \langle \beta | \alpha \rangle = \langle \alpha | \beta \rangle^* \]

Norm (length)
\[ ||\alpha|| = \sqrt{\langle \alpha | \alpha \rangle} \]

A set of vectors is orthonormal if
\[ \langle \alpha_i | \alpha_j \rangle = \delta_{ij} \]

W. r. t. an orthonormal basis
\[ \langle \alpha | \beta \rangle = \sum_{i=1}^{n} a_i^* b_i \]
\[ \langle \alpha | \alpha \rangle = \sum_{i=1}^{n} |a_i|^2 \]

The components are \[ a_i = \langle e_i | \alpha \rangle \] with \[ e_i \] the basis vectors.

Schwarz inequality

Generalization of angle between two vectors with
\[ |\cos \theta| \leq 1 \]
\[ |\langle \alpha | \beta \rangle| \leq ||\alpha|| ||\beta|| \]

Linear transformations (related to operators
\[ \frac{d}{dx} \] \[ T \]
\[ T \text{ transforms each vector } |\alpha\rangle \rightarrow |\alpha'\rangle = T|\alpha\rangle \]
\[ T(a|\alpha\rangle + b|\beta\rangle) = aT|\alpha\rangle + bT|\beta\rangle \]

If you know what \( T \) does to each basis vector
\[ T|e_i\rangle = T_1|e_1\rangle + T_2|e_2\rangle + \ldots + T_n|e_n\rangle \]
\[ T |e_j> = \sum T_{ij} |e_i> \]

For arbitrary \( i \), \( |\alpha> = \sum a_i |e_i> \)

\[ T |\alpha> = \sum a_j T |e_j> = \sum \sum T_{ij} a_j |e_i> \]

\( T \) takes a vector with components \( a_j \)

and gives a new vector \( a'_i = \sum T_{ij} a_j \)

This is matrix multiplication. Think \( T \) as a matrix:

\[
T = \begin{pmatrix}
T_{11} & T_{12} & \cdots & T_{1n} \\
T_{21} & T_{22} & \cdots & T_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
T_{n1} & T_{n2} & \cdots & T_{nn}
\end{pmatrix}
\]

\( n \times n \) square matrix

\[
\begin{pmatrix}
a'_1 \\
\vdots \\
a'_n
\end{pmatrix}
= \begin{pmatrix}
T_{11} & T_{12} & \cdots & T_{1n} \\
T_{21} & T_{22} & \cdots & T_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
T_{n1} & T_{n2} & \cdots & T_{nn}
\end{pmatrix}
\begin{pmatrix}
a_1 \\
\vdots \\
a_n
\end{pmatrix}
\]

Sum of two transformations \( S + T \)

\[
(S + T) |\alpha> = S |\alpha> + T |\alpha>
\]

\[
U = S + T \Rightarrow U_{ij} = T_{ij} + S_{ij}
\]
Product of two transformations: \( S \circ T \)

First do \( S \) to \( \hat{T} \) and then do \( S \) on resulting vector.

\[
|\alpha\rangle \rightarrow |\alpha'\rangle = \hat{T}|\alpha\rangle \rightarrow |\alpha''\rangle = S|\alpha'\rangle
\]

\[
|\alpha''\rangle = \sum_j S_{ij} \hat{T}_{jk} a_k = \sum_k U_{ik} a_k
\]

**Matrix multiplication**

\[
U = S \circ \hat{T} = \sum_j S_{ij} \hat{T}_{jk}
\]

**Transpose** (first a single vector)

\[
T^T = \begin{pmatrix}
T_{11} & T_{12} & \cdots & T_{1n} \\
T_{21} & T_{22} & \cdots & T_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
T_{n1} & T_{n2} & \cdots & T_{nn}
\end{pmatrix}
\]

Interchange the subscripts: \( \hat{T}_{ij} = T_{ji} \)

\[
\tilde{\alpha} = \alpha {\times n \times 1 \text{ row matrix}}
\]

\[
\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)
\]

A matrix is **symmetric** if \( \hat{T} = T \)

**anti-symmetric** if \( \hat{T} = -T \)

**Complex conjugate** of a matrix is complex conjugate of each element.
$$T^* = \left( \begin{array}{cccc} T_1^* & T_2^* & \cdots & T_n^* \\ T_1^* & T_2^* & \cdots & T_n^* \\ \vdots & \vdots & \ddots & \vdots \\ T_1^* & T_2^* & \cdots & T_n^* \end{array} \right)$$

Hermitian conjugate or adjoint of a matrix is transposed conjugate:

$$T^+ = S T^x = \left( \begin{array}{cccc} T_{11}^* & T_{12}^* & \cdots & T_{1n}^* \\ T_{21}^* & T_{22}^* & \cdots & T_{2n}^* \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1}^* & T_{n2}^* & \cdots & T_{nn}^* \end{array} \right)$$

$$a^+ = (a_1^*, a_2^*, \ldots, a_n^*)$$

Matrix is Hermitian if $$T^+ = T$$

Skew Hermitian if $$T^+ = -T$$

Inner product of two vectors:

$$\langle x | p \rangle = \frac{a^* b}{\equiv}$$

Careful of order $$S T = T S$$

Commutator

$$[S, T] = ST - TS$$

$$(ST)^+ = T^+ S^+ \neq S^+ T^+$$
Unit matrix: \[ I_{ij} = \delta_{ij}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

**Inverse**

\[ I^{-1} \cdot I = I = I \cdot I^{-1} \]

A matrix has an inverse if its determinant is non-zero.

\[ (ST)^{-1} = T^{-1} \cdot S^{-1} \]

Order switched.

A matrix is unitary if \[ U^* = U^{-1} \]
Example

Problem 2.26

(a) Sketch $V$

\[ V = -\alpha \left[ \delta(x+a) + \delta(x-a) \right] \]

(b) How many bound states does it have?

Think of two isolated $\delta$ functions in time $-a \to \infty$

\[ \psi_1(x) \quad \text{and} \quad \psi_2(x) \]

Pot. $V$ is symmetric $x \to -x$ so can find solutions $\psi$ even

$\psi(x) = \psi(-x)$

$\psi(x) = \psi_1(x) \psi_2(x)$

$\psi \text{ odd}$

$\psi(x) = -\psi(-x)$

$\psi = \psi_1(x) - \psi_2(x)$

Thus expect at most two bound states. Since each isolated $\delta$ function has only one bound state.
Expect even state to have lower E than odd state because odd state has an extra node.

If delta function is very close together in limit with strength \(2\pi\). In liquid state, odd solution as \(a \to 0\)

Thus even solution always exists, odd solution may not exist if \(a\) is too small.

Look for even solution \(E < 0\)

\[
\psi = A \left[ e^{\frac{2\pi x}{a}} + e^{-\frac{2\pi x}{a}} \right] \quad -a < x < a
\]

\[
\psi = B e^{-\pi x} \quad x > a
\]

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = EV
\]

\[
\Rightarrow \quad \frac{1}{\hbar} = \sqrt{-\frac{2mV}{\hbar^2}} \quad \text{for} \quad E < 0
\]

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = EV
\]
Integrate eq
\[-\frac{\hbar^2}{2m} \left( \frac{d^2 \psi}{dx^2} + \frac{\partial^2 \psi}{\partial x^2} \right) - \alpha \psi(a) = 0\]
\[a + c \quad a - c\]
\[-\frac{\hbar^2}{2m} \left[ -\frac{\partial^2 \psi}{\partial x^2} \right] + \int \psi \psi^* dx = A \left[ e^{2\pi \alpha} - e^{-2\pi \alpha} \right] = \alpha \psi\]
\[-\int \psi \psi^* dx \quad = \frac{2m \alpha}{\hbar^2} \left( e^{2\pi \alpha} + e^{-2\pi \alpha} \right) = \frac{4m \alpha}{\hbar^2} \]
\[\frac{\hbar^2}{2m} \left[ e^{2\pi \alpha} + e^{-2\pi \alpha} \right] = 2m \alpha \frac{\alpha}{\hbar^2}\]
\[\int \left[ 1 + e^{-2\pi \alpha} \right] = \frac{\alpha}{\hbar^2}\]

Can be solved numerically.

Given \( J_c \Rightarrow E = -\frac{\hbar^2}{2m} J_c \)

Note this equation always has a solution.

For odd solution \( \psi \rightarrow A \left[ e^{2\pi \alpha} - e^{-2\pi \alpha} \right] \)

As before except now
\[\int \left[ 1 - e^{2\pi \alpha} \right] = \frac{\alpha}{\hbar^2}\]

This does not have a solution if \( a \) is too small, i.e., left hand side always greater than right hand side.