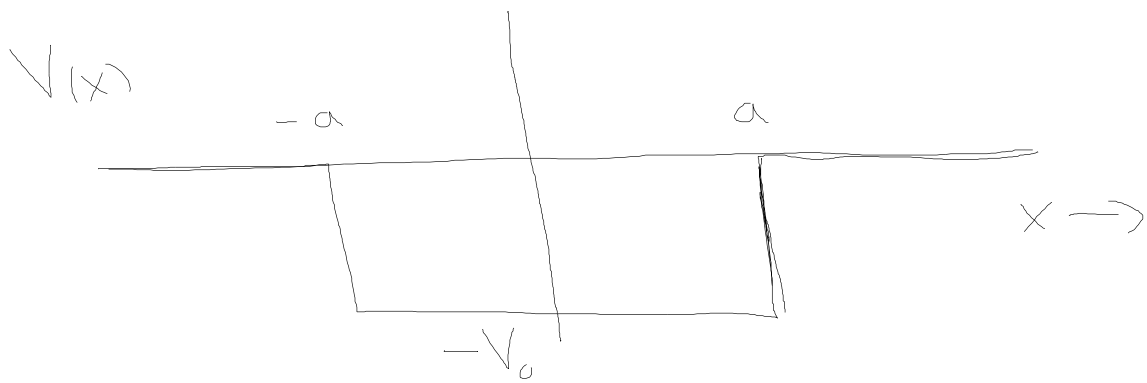
Lecture # 14

Example of weakly bound state

Finite square well: a and $V_0 \to$ small



$$\psi(x) = \begin{cases} F e^{-\kappa x} & x > a \\ D \cos lx & 0 < x < a \\ \psi(-x) & x < 0 \end{cases}$$

$$\kappa = \sqrt{-2mE}/\hbar \qquad l = \sqrt{2m(E+V_0)}/\hbar$$

Expect $la \ll 1$ because we have week, narrow well.

$$\cos lx \simeq 1 - (lx)^2/2$$

$$\psi'/\psi \Big|_{x=a} \simeq -l^2 a \,/\, [1 - (la)^2/2] \simeq -l^2 a$$

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Outside $\psi = F e^{-\kappa x}$

$\simeq F(1 - \kappa \cancel{x})$

$\psi' / \psi \simeq -\kappa$

$\kappa = l^2 a$

Expect $l \simeq \sqrt{2mV_0} / \hbar$

since E is going to be $\simeq 0$

$$\kappa^2 = -\frac{2mE}{\hbar^2} = l^4 a^2$$

$$E = -\frac{\hbar^2}{2m} \left[\frac{2mV_0}{\hbar^2}\right]^2 a^2$$

$$\boxed{E = -\frac{2m}{\hbar^2} V_0^2 a^2}$$

Example: 1 dim model of deuteron

$E = -2.2$ MeV binding E

$m = \frac{1}{2} \, 939 \, \frac{\text{MeV}}{c^2}$

reduced mass

$\hbar c = 197.33$ MeV-Fm

$a \simeq 1 \text{ Fm} = 10^{-15}$ meters

$$-2.2 \text{ MeV} = -\frac{939 \text{ MeV}}{(\hbar c)^2} V_0^2 (1 \text{Fm})^2$$

$$\sqrt{\frac{(\hbar c)^2}{a^2} \frac{2.2}{939}} = V_0$$

Solve for well depth

$$\left(-\frac{\hbar^2 E}{2 m a^2} \right)^{1/2} = V_0$$

$$\left(\frac{-(\hbar c)^2 \quad E}{2 \ m c^2 \quad a^2} \right)^{1/2} = V_0$$

$$\left(\frac{(197.33)^2 \quad 2.2}{(1 \text{Fm})^2 \quad 939} \right)^{1/2} = \boxed{9.55 \text{ MeV} = V_0}$$

We will find we need a deeper well in 3 dim.

$$\kappa = \frac{\sqrt{2.2 \ 939}}{197.33} = 0.230 \text{ Fm}^{-1}$$

Expect size of deuteron $\sim 1/\kappa$
$= 4.3 \text{ Fm} > a \simeq 1 \text{ Fm}$

Normalize

$$\psi = \begin{cases} A \cos lx & |x| < a \\ A \cos la \; e^{\kappa a} \; e^{-\kappa |x|} & |x| > a \end{cases}$$

Wave function is cont. at $x = a$

Energy insures derivative is cont.

$$\int_{-\infty}^{\infty} dx\, \psi^* \psi = 1 = 2 \int_0^{\infty} dx\, \psi^* \psi$$

$$= 2 A^* A \left[\int_0^a \cos^2 lx \, dx + \int_a^{\infty} \cos^2 la \; e^{2\kappa a} e^{-2\kappa x} \, dx \right]$$

assume weak bound state $la < 1$

$$\cos lx \simeq 1 - (lx)^2/2 \simeq 1 \qquad \text{for } x \le a$$

$$1 = 2 A^* A \left[\int_0^a dx + e^{2\kappa a} \frac{1}{2\kappa} e^{-2\kappa a} \right]$$

$$1 = 2 A^* A \left[a + \frac{1}{2\kappa} \right]$$

$$\boxed{A = \left[2a + \kappa^{-1} \right]^{-1/2}}$$

Prob inside well

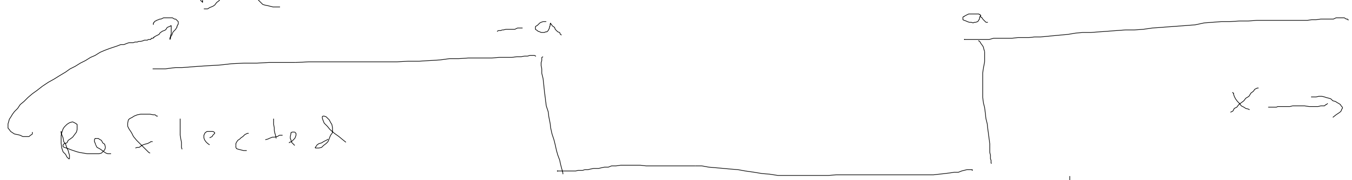
$$P_{in} = \int_{-a}^{a} \psi^* \psi \, dx = 2 A^* A \int_0^a dx$$

$$= 2a A^* A = \frac{2a}{2a + \frac{1}{\kappa}} = \frac{2\text{ Fm}}{2\text{ Fm} + 4.35\text{ Fm}}$$

$$\boxed{P_{in} = .31} \qquad P_{out} = 1 - P_{in} = 0.69$$

Scattering States

Incident $Ae^{ikx} + Be^{-ikx}$ $\qquad$ $C \sin lx + D \cos lx$ $\qquad$ Transmitted Fe^{ikx}



$E > 0$ so $e^{-\varkappa |x|} \to e^{ikx}$

Match ψ, $\frac{d\psi}{dx}$ at $x = a$ and $x = -a$

4 equations: Find C, D, B, F given A

$$F = A \frac{e^{-2ika}}{\cos 2la - i \frac{\sin 2la}{2ka}(k^2 + l^2)}$$

see problem 2.31

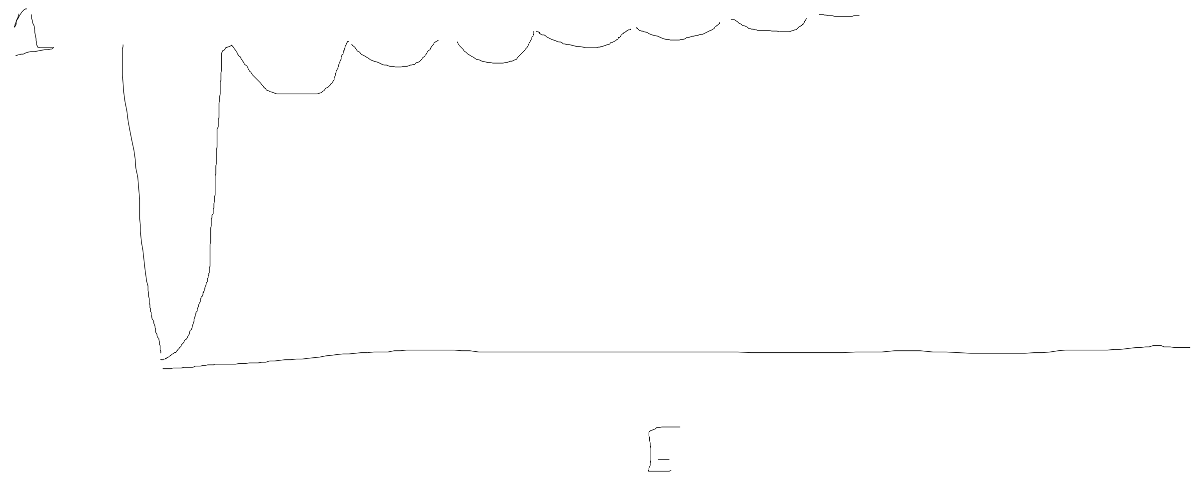
Fraction of wave transmitted

$$T \equiv F^* F / A^* A$$

$$T = \left[1 + \frac{V_0^2}{4E(E+V_0)} \sin^2 \left(\frac{2a}{\hbar} \sqrt{2m(E+V_0)} \right) \right]^{-1}$$

Reflected prob.

$R = 1 - T$

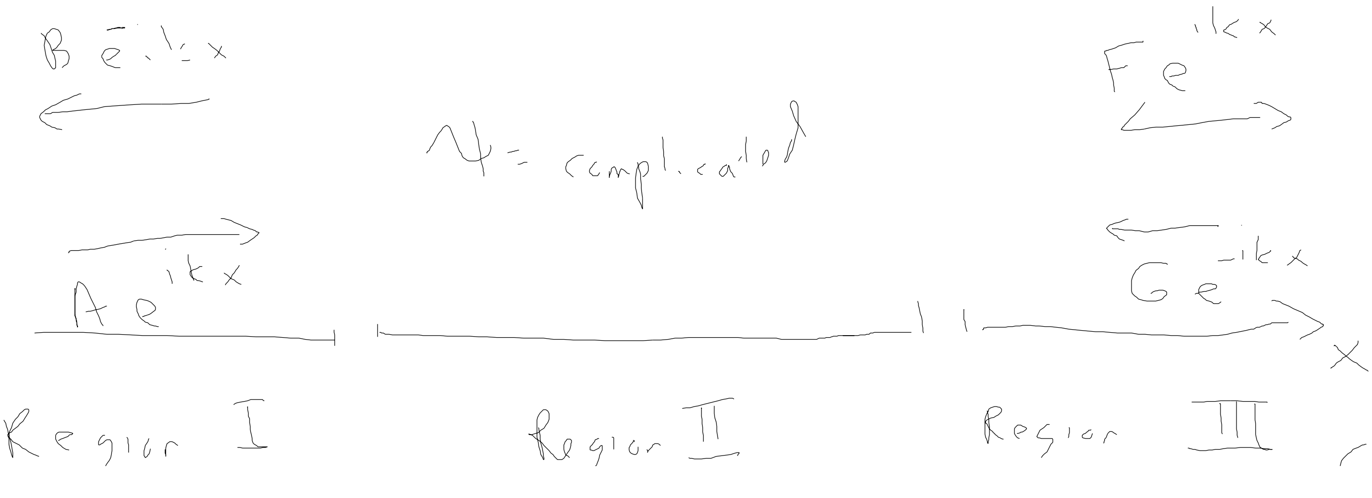


T does $\to 1$ as $E \to \infty$
But $\neq 1$ for most finite E

Reflection from an attractive well is a quantum phen. In classical mech. $T = 1$ for all $E > 0$

Scattering Matrix

Consider a localized pot. which is only $\neq 0$ in region II



Incident waves A, G produce reflected and transmitted waves B, F

Match B.C. at two interfaces

$$\begin{bmatrix} B \\ F \end{bmatrix} = \begin{bmatrix} \underline{\underline{S}} \end{bmatrix} \begin{bmatrix} A \\ G \end{bmatrix}$$

Scattering matrix $\underline{\underline{S}}$ 2×2 complex matrix

$$\underline{\underline{S}} = \begin{bmatrix} S_{11}(E) & S_{12}(E) \\ S_{21}(E) & S_{22}(E) \end{bmatrix}$$

S is a very general concept and exists for any short range pot

Example reflection coef from left no wave from right $(G = 0)$

$$R_l = \frac{|B|^2}{|A|^2} = |S_{11}|^2$$

$$T_l = |F|^2 / |A|^2 = |S_{21}|^2$$

For scattering from right $A = 0$

$$R_r = |F|^2 / |G|^2 = |S_{22}|^2$$