Lecture #36 Perturbation Theory

Perturbation theory is a systematic procedure for obtaining approximate solutions to the perturbed problem by building on known exact solutions to unperturbed case.

Unperturbed

\[ H_0 \Psi_0^n = E_n^0 \Psi_0^n \]

Assume both \( E_n^0 \) and \( \Psi_0^n \) are known

\[ \langle \Psi_0^n | \Psi_0^n \rangle = 5_{rn} \]

Like to solve \( H \Psi_n = E_n \Psi_n \)

\[ H = H_0 + \chi H' \]

Method works best if \( \chi H' \) is small. \( \chi \) is a device for counting powers of small perturbations \( H' \) at each order. \( \chi = 1 \).

Example

\[ \frac{p^2}{2m} \Rightarrow -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \]

Nonrel. Kinetic energy.

\[ \text{Should have used } \sqrt{p^2c^2 + m^2c^4} - mc^2 \]
\[ p^2 - \frac{p^4}{2 \alpha m^8 \hbar^2 c^2} \]

Thus a relativistic correction to the nonrel. Hamiltonian is

\[ H' = - \frac{\hbar^2}{8 m^3 c^2} \frac{\partial^4}{\partial x^4} \]

In general this is a very small correction because \( p \ll m c \) i.e. \( \frac{v}{c} \ll 1 \)

\[ H^0 = \frac{p^2}{2m} + V \quad , \quad H' = - \frac{p^4}{8 m^3 c^2} \]

But pert. theory works for lots of problems

Expand Wave function and eigenvalue in power series in \( \lambda \)

\[ \psi_n = \psi_n^0 + \lambda \psi_n^1 + \lambda^2 \psi_n^2 + ... \]

\[ E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + ... \]

\[ [H^0 + \lambda H'] [\psi_n^0 + \lambda \psi_n^1 + \lambda^2 \psi_n^2 + ... ] = \]

\[ \lambda^2 E_n^2 \]
\[
\begin{align*}
&\left[ E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \cdots \right] \left[ \psi_n^0 + \lambda \psi_n^1 + \lambda^2 \psi_n^2 + \cdots \right] \\
&= \text{equate power of } \lambda \text{ (and set } \lambda = 1) \\\n&= H_n^0 \psi_n^0 = E_n^0 \psi_n^0 \text{ unpert. problem} \\
&= H_n^0 \psi_n^1 + H_n^1 \psi_n^0 = E_n^0 \psi_n^1 + E_n^1 \psi_n^0 \quad \text{(A)} \\
&= H_n^0 \psi_n^2 + H_n^1 \psi_n^1 = E_n^0 \psi_n^2 + E_n^1 \psi_n^1 + E_n^2 \psi_n^0 \quad \text{(B)} \\
\end{align*}
\]

Take inner product of (A) with \( \psi_n^0 \) and set \( \lambda = 1 \)

\[
\begin{align*}
&\langle \psi_n^0 | H_0^0 \psi_n^0 \rangle + \langle \psi_n^0 | H_1^1 \psi_n^0 \rangle \\
&= E_n^0 \langle \psi_n^0 | \psi_n^1 \rangle + E_n^1 \langle \psi_n^0 | \psi_n^1 \rangle + E_n^2 \langle \psi_n^0 | \psi_n^0 \rangle \\
&\langle \psi_n^0 | \psi_n^1 \rangle = E_n^0 \langle \psi_n^0 | \psi_n^1 \rangle \quad \text{let } H_0^0 \text{ act to left.} \\
&\langle \psi_n^0 | \psi_n^0 \rangle = 1 \\
&\begin{array}{c}
E_n^1 = \langle \psi_n^0 | H_1^1 \psi_n^0 \rangle \\
\end{array}
\]

1st order shift in energy. This is just expectation of perturbation in unperturbed wave function.

To find 1st order correction to wave function.


rewrite \(^{(1)}\)

\[(H^0 - E_n) \Psi_n = - (H' - E_n') \Psi_n\]

Expand \(\Psi_n = \sum \sum_{m \neq n} c_{m,n} \Psi_m \Psi_n\)

In general, we will mix a little bit of all of the other states \(m \neq n\) into \(\Psi_n\).

\[\sum (H^0 - E_n) c_{m,n} \Psi_m = - (H' - E_n') \Psi_n\]

Take inner product with \(\Psi\)

\[\sum (E_n^c - E_n^c) c_{m,n} \langle \Psi_n | \Psi_n \rangle = - \langle \Psi_n | H' | \Psi_n \rangle + E_n' \langle \Psi_n | \Psi_n \rangle\]

Assume \(m \neq n\) Her. \(\langle \Psi_n | \Psi_n \rangle = \delta_{m,n}\)

\[(E_n^c - E_n^c) c_{m,n} = - \langle \Psi_n | H' | \Psi_n \rangle + E_n' \langle \Psi_n | \Psi_n \rangle\]

or

\[c_{m,n} = \frac{\langle \Psi_n | H' | \Psi_n \rangle}{E_n^c - E_n^c}\]

\[\Psi_n = \sum_{m \neq n} \frac{\langle \Psi_n | H' | \Psi_n \rangle}{E_n^c - E_n^c} \Psi_m\]

Note, normalization \(\langle \Psi_n | \Psi_n \rangle = 0\)
So total wave function \( \langle \psi_n^0 + \psi_n^1 | \psi_n^0 \rangle = 1 \).

Possible problem if energy eigenvalues are degenerate. They denumerate can vanish.

Assume unperturbed spectrum is nondegenerate for now. Then \( m \neq n \) in \( \sin \) is enough to keep energy from vanishing.

Example Problem 6.1

Square well with \( S \) for \( 0 \leq x \leq a \)

\[ H' = \alpha \delta(x-a/2) \quad \text{assume } x \text{ small} \]

Need unperturbed wave function and energies

\[ E_n^0 = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \]

\[ \psi_n^0(x) = \sqrt{\frac{2}{a}} \sin \left( \frac{n \pi x}{a} \right) \]

\[ E_n = \langle \psi_n^0 | H' | \psi_n^0 \rangle \]

\[ = \frac{2}{a} \sum_0^\infty \sin^2 \left( \frac{n \pi x}{a} \right) \delta(x-a/2) \]

\[ = \frac{2}{a} \sin^2 \left( \frac{n \pi}{2} \right) \]

\[ \text{if } n \text{ odd } 5 \]