# Lecture #33 Solids

In a solid a few loosley bound outermost valence electrons in each atom become detached and roam throughout the material.

Simple model $\Rightarrow$ Free electron gas

$$V = \begin{cases} 0 & \text{inside} \\ \infty & \text{outside metal} \end{cases}$$

Consider a box of length $l_x, l_y, l_z$

Solve Schrod. Eq.

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

in Cartesian coordinates

$$\psi_{n_x n_y n_z} = \sqrt{\frac{8}{l_x l_y l_z}} \sin\left(\frac{n_x \pi}{l_x}x\right)\sin\left(\frac{n_y \pi}{l_y}y\right)\sin\left(\frac{n_z \pi}{l_z}z\right)$$

with $n_x, n_y, n_z = 1, 2, 3 \ldots$

$$E = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{l_x^2} + \frac{n_y^2}{l_y^2} + \frac{n_z^2}{l_z^2}\right) = \frac{\hbar^2 k^2}{2m}$$

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Wave vector $\vec{k} = (k_x, k_y, k_z)$

$$k^2 = \pi^2 \left( \frac{n_x^2}{l_x^2} + \frac{n_y^2}{l_y^2} + \frac{n_z^2}{l_z^2} \right)$$

Fill energy levels $\lambda$ at a time (with $\lambda = 2$ spin degeneracy) till some maximum $E_{max} = E_F$

$$E \leq E_F \quad \text{Fermi energy}$$

$$E_F = \frac{\hbar^2}{2m} k_F^2 \quad \text{defines Fermi momentum } \hbar k_F$$

<u>Counting states:</u> Total of $N_{tot}$ electrons in system of volume $V$

$$V = l_x l_y l_z$$

Number density of electrons

$$\rho = \frac{N_{tot}}{V} \quad \text{(Units particles per } cm^3\text{)}$$

Interested in $N_{tot}$ very large $\sim 10^{24}$

Consider cube $l_x = l_y = l_z = l$

Fill all states $n_x, n_y, n_z$ such that

$$n_x^2 + n_y^2 + n_z^2 \leq n_{last}^2$$

$$n_{last}^2 = k_F^2 \left(\frac{l^2}{\pi^2}\right)$$

Count

$n_x^2 + n_y^2 + n_z^2 \leq n_{last}^2$ $\quad |\vec{n}| \leq n_{last}$

$$N_{tot} = \sum_{n_x n_y n_z = 1} \lambda \approx \frac{\lambda}{8} \int d^3 n$$

only positive $n$

$$k_x = \frac{\pi}{l} n_x \Rightarrow d^3 k = \frac{\pi^3}{l^3} d^3 n$$

$k \leq k_F$

$$N_{tot} = \frac{\lambda \; l^3}{8 \; \pi^3} \int d^3 k \qquad l^3 = V$$

$$\rho = \frac{N_{tot}}{V} = \frac{\lambda}{8\pi^3} \int^{k_F} d^3 k$$

$$\rho = \frac{\lambda}{8\pi^3} \int_0^{k_F} k^2 \, dk \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$\rho = \frac{\lambda \; 4\pi}{8 \; \pi^3} \left(\frac{k_F^3}{3}\right) = \frac{\lambda}{6\pi^2} k_F^3 = \frac{k_F^3}{3\pi^2}$$

Fermi momentum $p_F = \hbar k_F$

$$k_F = (3\rho\pi^2)^{1/3}$$

grows with density.

Total energy

$$E_{tot} = \lambda \sum_{n_x n_y n_z}^{n_x^2+n_y^2+n_z^2 \leq n_{last}^2} \frac{\hbar^2}{2m} \frac{\pi^2}{l^2} (n_x^2+n_y^2+n_z^2)$$

$$= \lambda V \int^{k_F} \frac{d^3k}{8\pi^3} \frac{\hbar^2}{2m} k^2$$

$$= \lambda \frac{4\pi V}{8\pi^3} \int_0^{k_F} k^2 dk \frac{\hbar^2 k^2}{2m}$$

$$E_{tot} = \frac{\lambda V}{2\pi^2} \left(\frac{k_F^5}{5}\right) \frac{\hbar^2}{2m}$$

$$= \left(\frac{\lambda \hbar^2}{20\pi^2 m}\right) V (3\pi^2)^{5/3} \left(\frac{N_{tot}}{V}\right)^{5/3} \qquad \lambda = 2$$

$$E_{tot} = \frac{\hbar^2 (3\pi^2)^{5/3}}{10\pi^2 m} N_{tot}^{5/3} V^{-2/3}$$

$$\frac{dE_{tot}}{dV} = -\frac{2}{3} \frac{E_{tot}}{V} = -P$$

If you compress the gas the energy rises and the electrons press harder against the walls of the box

$$\boxed{P = \frac{2}{3}\frac{E_{tot}}{V} = \frac{\hbar^2 k_F^5}{10\pi^2 m}}$$

$$P \propto V^{-5/3} \quad \text{since } k_F \propto V^{-1/3}$$

<u>Metalic Hydrogen</u>

At normal density and pressur solid H is a mollecular compound with electrons confined into $H_2$ mollecules.

However as you increase the pressur you can induce a transition to a metalic state with the electrons forming a more or less free <u>Fermi sea</u> inside the material. Like last electrons in a sodium crystal.

This takes ~ million atmospheres

Comp. between Coulomb energy $\sim \frac{1}{r} \sim V^{-1/3}$ attracting electron into molecule and Fermi energy $\sim \frac{\hbar^2 k_F^2}{2m} \sim V^{-2/3}$

at high density Fermi energy wins and you have nearly free Fermi gas of electrons. This is called pressure ionization.

Core of Jupiter is metalic H.

Our Sun will event. run out of H fuel in ~ 5 billion years and puff up to become red giant then burn He to C + O and run out of He. Finally without nuclear burning to keep core hot it will collapse all the way to a <u>white dwarf</u>

White dwarf is a planet sized object of ~ solar mass which is supported against gravity by Fermi pressure of electrons.

Example: Sirius B (companion to Sirus A) has $M = 1.053 \pm .028 \; M_\odot$ and $r = .0074 \pm .0006 \; r_\odot$