11/13/98

Lecture #31 Identical Particles

Two particle systems:
For single particle $\Psi(\vec{r}, t)$ is a function of the spatial coordinates $\vec{r}$ (ignore spin for a second)

Wave function for two particles is a function of $\vec{r}_1$ and $\vec{r}_2$

$$\Psi(\vec{r}_1, \vec{r}_2, t)$$

Prob. to find particle 1 within Δr_1^3 of $\vec{r}_1$ <u>and</u> particle 2 within Δr_2^3 of $\vec{r}_2$ is

$$P_{12}(\vec{r}_1, \vec{r}_2) = \Psi^*(\vec{r}_1, \vec{r}_2, t)\, \Psi(\vec{r}_1, \vec{r}_2)\, \Delta r_1^3\, \Delta r_2^3$$

Normalization $\int d^3r_1 \int d^3r_2\ \Psi^* \Psi = 1$

Prob. to find particle 1 within $(\Delta r_1)^3$ of r_1 and particle 2 anywhere is

$$P_1(\vec{r}_1) = \int d^3r_2\ \Psi^*(\vec{r}_1, \vec{r}_2)\, \Psi(\vec{r}_1, \vec{r}_2)\ (\Delta r_1)^3$$

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}_1, \vec{r}_2, t) = H \Psi$$

$$H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2)$$

time indep. solutions

$$\Psi(\vec{r}_1, \vec{r}_2, t) = \psi(\vec{r}_1, \vec{r}_2)\, e^{-iEt/\hbar}$$

$$-\frac{\hbar^2}{2m}\nabla_1^2 \psi - \frac{\hbar^2}{2m_2}\nabla_2^2 \psi + V\psi = E\psi$$

Note in general there are complicated correlations between the motions of the two particles. Example two electrons in a He atom repel each other so don't find them close together.

$$\psi(r_1, r_2) \to \text{small for small } |\vec{r}_1 - \vec{r}_2|$$

As a simple case suppose interaction between two particles is small then expect ψ to be at least approx. a direct <u>product</u>

$$\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\, \psi_b(\vec{r}_2)$$

This assumes we can tell the two particles apart! Otherwise it does not make sense to say we know particle 1 is in state a and particle 2 is in state b.

Note in classical mech. you can always tell the particles apart, at least in principle. Paint one of them red or notch one of their ears ... or hire a detective to follow one of them around.

In QM situation is very different! Electrons only have a very few properties: mass, charge, spin, magnetic moment and all of these are exactly identical for all electrons. There is no room to paint an electron red or notch its ear because it does not have a color and it does not have ears.

Remember color arises because of the different ways electrons are bound into different substances. Bind it one way and an electron can be part of a red material. Bind it a different way and that same electron is part of a green material.

Quantum mechanics accomodates the existence of particles that are indistinguishable in principle.

If I don't know that one is in state a and two is in state b try a symmetrized linear combination

$$\psi_{\pm}(\vec{r}_1, \vec{r}_2) = A\left[\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)\right]$$

Bosons should by symmetric under interchange of 1 and 2 (plus sigh)

Fermions are antisymmetric under interchange (allways use minus sign)

Spin stattistics theorem

- all integer spin particles are bosons
- all half-integer spin particles are fermions

Photons and mesons are bosons. Electrons, protons, neutrons, quarks are fermions.

Need relativistic quantum field theory to prove spin statistics theorm. However it is a fundamental principle of nature.

Two identical fermions can not occupy same state. If $\psi_a = \psi_b$ then

$$\psi_-(r_1, r_2) = A\left[\psi_a(r_1)\psi_a(r_2) - \psi_a(r_2)\psi_a(r_1)\right]$$
$$= 0$$

This is the <u>Pauli exclusion principle</u>

$$\psi(\vec{r}_1, \vec{r}_2) = \pm\, \psi(\vec{r}_2, \vec{r}_1)$$

We require wave function to be antisymmetric for the interchange of all of the coordinates of any two fermions (and symmetric for bosons).

<u>He atom</u> $Z = 2$

$$H = \left\{-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{2e^2}{4\pi\epsilon_0 r_1}\right\} + \left\{-\frac{\hbar^2\nabla_2^2}{2m} - \frac{2e^2}{4\pi\epsilon_0 r_2}\right\}$$
$$+ \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec{r}_1 - \vec{r}_2|}$$

If ignore last term

$$\psi(\vec{r}_1, \vec{r}_2) = \psi_{n\ell m}(\vec{r}_1)\,\psi_{n'\ell'm'}(\vec{r}_2)$$

$$-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{2e^2}{4\pi\epsilon_0 r_1}\psi_{n\ell m}(r_1) = E_n(Z=2)\,\psi_{n\ell m}(r_1)$$

$$E_n(Z=2) = Z^2 E_n(Z=1) = -(13.6\text{ eV})\frac{4}{n^2}$$

$$E_{tot} = -13.6\text{ eV}\left(\frac{4}{n^2} + \frac{4}{n'^2}\right)$$

For ground state $n = n' = 1$

$$E_{tot} = -8(13.6\text{ eV}) = -109\text{ eV}$$

This is greater than exp. -78.975 eV because we ignored e – e repulsion

Note total wave function must be antisym. under interchange of all coordinates (space + spin) of electron #1 and #2

Use antisymmetric spin state (spin singlet)

$$|00\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

This is odd under $1 \leftrightarrow 2$

Ground state He wave func.

$$\Psi(r_1, r_2) = \psi_{000}(r_1)\,\psi_{000}(r_2)\,\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

in approx. of ignor. $e^2/|\vec{r}_1 - \vec{r}_2|$