11/11/98

Lecture #30 Addition of angular momentum

Consider two spin 1/2 electrons

$$S_z^{(1)} \chi_1 = \hbar m_1 \chi_1$$

$$S_{(1)}^2 \chi_1 = \hbar^2 s_1(s_1+1) \chi_1$$

with $s_1 = 1/2 = s_2$

Likewise $S_z^{(2)} \chi_2 = \hbar m_2 \chi_2$

$$S_{(2)}^2 \chi_2 = \hbar^2 s_2(s_2+1) \chi_2$$

Consider total spin angular momentum

$$S_z = S_z^{(1)} + S_z^{(2)}$$

and $S^2 = S_{(1)}^2 + S_{(2)}^2$

$$\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$$

Adding S_z is easy. Total z comp. is just sum of individual z comp.

$$S_z \chi_1 \chi_2 = (S_z^{(1)} + S_z^{(2)}) \chi_1 \chi_2 = \hbar(m_1+m_2)\chi_1\chi_2$$

Would like to make an eigenstate of S^2

Consider four possible states

spin	1 2			
	↑↑	↑↓	↓↑	↓↓
m =	1	0	0	-1

$\uparrow\uparrow = \chi_{1+} \chi_{2+}$ $\qquad$ $\downarrow\uparrow = \chi_{1-} \chi_{2+}$

$$S^2 = S_{(1)}^2 + S_{(2)}^2 + 2\,\vec{S}_{(1)}\cdot\vec{S}_{(2)}$$

$$\vec{S}_{(1)}\cdot\vec{S}_{(2)} = S_{(1)}^x S_{(2)}^x + S_{(1)}^y S_{(2)}^y + S_{(1)}^z S_{(2)}^z$$

$$S_x^{(1)} = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad S_x^{(1)}\downarrow = \frac{\hbar}{2}\uparrow \qquad S_x^{(1)}\uparrow = \frac{\hbar}{2}\downarrow$$

ie $\frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$S_y^{(1)} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_y^{(1)}\uparrow = i\frac{\hbar}{2}\downarrow \qquad S_y^{(1)}\downarrow = -i\frac{\hbar}{2}\uparrow$$

Note $S_{(1)}^2$, $S_{(2)}^2$ are easy

$$S_{(1)}^2\uparrow = \hbar^2\frac{3}{4}\uparrow \qquad S_{(1)}^2\downarrow = \hbar^2\frac{3}{4}\downarrow$$

$$S^2 = \frac{3}{2}\hbar^2 + 2\,\vec{S}_{(1)}\cdot\vec{S}_{(2)}$$

We would like to make an eigenstate of $\vec{S}_1\cdot\vec{S}_2$

Consider $\uparrow\downarrow$

$$S_x^{(1)} S_x^{(2)}\uparrow\downarrow = \left(\frac{\hbar}{2}\right)^2\downarrow\uparrow$$

$$S_y^{(1)} S_y^{(2)}\uparrow\downarrow = \left(\frac{\hbar}{2}\right)^2 i(-i)\downarrow\uparrow = \frac{\hbar^2}{4}\downarrow\uparrow$$

$$S_z^{(1)} S_z^{(2)}\uparrow\downarrow = -\frac{\hbar^2}{4}\uparrow\downarrow$$ Spins not flipped

$$\vec{S}_1\cdot\vec{S}_2\uparrow\downarrow = \frac{\hbar^2}{4}\left(2\downarrow\uparrow - \uparrow\downarrow\right)$$

Likewise $\vec{S}_1 \cdot \vec{S}_2 \downarrow\uparrow = \frac{\hbar^2}{4}(2\uparrow\downarrow - \downarrow\uparrow)$

So consider linear combination (normalized)

$$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

$$\vec{S}_1 \cdot \vec{S}_2 \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}\left[2\downarrow\uparrow - \uparrow\downarrow - 2\uparrow\downarrow + \downarrow\uparrow\right]$$

$$= -3\frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

Thus $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ is an eigenstate of $\vec{S}_1 \cdot \vec{S}_2$ with eigenvalue $-\frac{3}{4}\hbar^2$

$$S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2$$

$$= \frac{3}{2}\hbar^2 + 2 S_1 \cdot S_2$$

So $S^2 \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) = \left[\frac{3}{2}\hbar^2 + 2\left(-\frac{3}{4}\hbar^2\right)\right]\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$

$$= 0 \; \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

$\Rightarrow$ Therefore $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ is a spin <u>singlet</u> state $s = 0$ $m = 0$.

$$S_z \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) = 0 \; \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

$$S^2 \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) = 0 \; \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

Consider

$$\vec{S}_1 \cdot \vec{S}_2 \ \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}\Big[2\downarrow\uparrow - \uparrow\downarrow$$

$$+ 2\uparrow\downarrow - \downarrow\uparrow\Big] = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(\downarrow\uparrow + \uparrow\downarrow)$$

So

$$S^2 \ \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) = \left(\frac{3\hbar^2}{2} + 2\frac{\hbar^2}{4}\right)\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$

$$= 2\hbar^2 \ \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$

$$\big(S(S+1) \qquad S = 1$$

Thus $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$ is a spin <u>triplet</u> state with $S=1$ $m=0$

Easy to show

$$S^2 \ \uparrow\uparrow = 2\hbar^2 \ \uparrow\uparrow$$

and

$$S^2 \ \downarrow\downarrow = 2\hbar^2 \ \downarrow\downarrow$$

Thus the three states

	m
$\uparrow\uparrow$	1
$\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$	0
$\downarrow\downarrow$	-1

all have $S=1$ and $m = 1, 0, -1$

So from four states $\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$, $\downarrow\downarrow$

make three $S=1$ states and one $S=0$.

In general add two angular momenta
For example orbital and spin

$$\vec{j} = \vec{l} + \vec{s}$$

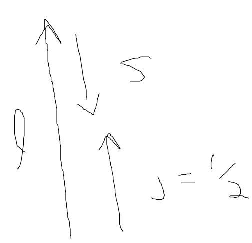
$\vec{s}$: spin of electron

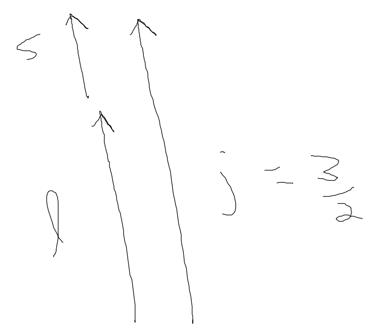
$\vec{l}$: orbital ang. momentum of electron around nucleus in H atom

Eigenvalues of $\vec{j}^{\,2}$ can run from $j = |l - s|$ to $l + s$

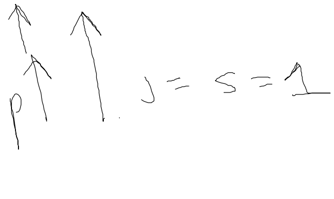
Example $l = 1$ $s = 1/2$ so $j = \frac{1}{2}$ or $\frac{3}{2}$

For $j = 3/2$ say m can run from $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}$ to $3/2$.





Deuterium has spin 1/2 neutron plus spin 1/2 proton in an $S = 1$ state
Spin dependence of N N force causes ground state to have spins alligned



<u>He atom</u> ground state has two electrons coupled to $S=0$

$e^- \uparrow \quad \downarrow e^- \quad - \quad S=0$

In general expect most spins to pair off to $S=0$.

In general state $|jm\rangle$ is a linear combination of Y_ℓ^m and χ_{m_s}

$$|jm\rangle = \sum_{m_\ell m_s} C^{\ell \frac{1}{2} j}_{m_\ell m_s m} Y_\ell^{m_\ell} \chi_{m_s}$$

Coef. is called Clebsch-Gordan coeff.
Sum over m_ℓ, m_s only runs over $m_\ell m_s$ such that $m_\ell + m_s = m$