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Lecture #29 Spin Cont.

$S_x = \frac{\hbar}{2}\sigma_x \qquad S_y = \frac{\hbar}{2}\sigma_y \qquad S_z = \frac{\hbar}{2}\sigma_z$

For spin ½ particle

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\sigma_i^{\dagger} = \sigma_i$ hermitian $\qquad i = x, y, z$

$\sigma_i^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}(1+1+1)$

$S^2 = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

<u>Eigenstates</u> of S_z

$\chi_+^{(z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigenvalue $+\frac{\hbar}{2}$

$\chi_-^{(z)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ eigenvalue $-\frac{\hbar}{2}$

$\underbrace{\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{S_z} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(-\frac{\hbar}{2}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ For example

$S_z\,\chi_-^{(z)} = s_z\,\chi_-^{(z)} \qquad s_z = -\hbar/2$

Both $\chi_+^{(z)}$ and $\chi_-^{(z)}$ are eigenstates of S^2 with eigenvalue $\frac{3}{4}\hbar^2 = s(s+1)\hbar^2$ with $s = ½$

Note $\chi_m^{(z)} = |\tfrac{1}{2}\ m\rangle$

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What are eigenstates of S_x?
Eigenvalues $\pm \hbar/2$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \quad b = a \quad \Rightarrow \quad \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For negative eigenvalue $a = -b$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Can expand any state in a complete set of eigenfunctions

$$\chi_+^{(z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_+ \chi_+^{(x)} + c_- \chi_-^{(x)}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_+ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_- \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \boxed{c_+ = c_- = \frac{1}{\sqrt{2}}}$$

(a) Prepare a system 100% in $\chi_+^{(z)}$
No uncert. in S_z.

(b) Measure S_x (i) Prob. to get $+\frac{\hbar}{2} = |c_+|^2 = \frac{1}{2}$
(ii) Prob. to get $-\frac{\hbar}{2} = |c_-|^2 = \frac{1}{2}$

(c) Assume you get $+\hbar/2$. Now know S_x with no uncert. Allready knew S_z so this violates uncert. principle since $[S_x, S_z] \neq 0$??

No! Instead measuring S_x disturbs the system and particle no longer is in eigenstate of S_z so uncert. in S_z is no longer zero.

Indeed after the S_x measurement that gives $+\hbar/2$ particle is in state

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This is a 50/50 mixture of $\chi_+^{(z)}$ and $\chi_-^{(z)}$ so if you measure S_z after measuring S_x instead of 100% spin up you will only see 50% spin up.

$\Rightarrow$ In QM measurement disturbs system. After measurement wave function in general changes to be in an eigenstate of the operator you measured.

<u>How to measure spin?</u>

Electron in a magnetic field
Spinning charge has a magnetic dipole moment μ

$$\vec{\mu} = \gamma \vec{S}$$

$$\gamma = \frac{e}{2m} g \qquad g = 2 + \text{small corrections}$$

Gyromagnetic ratio

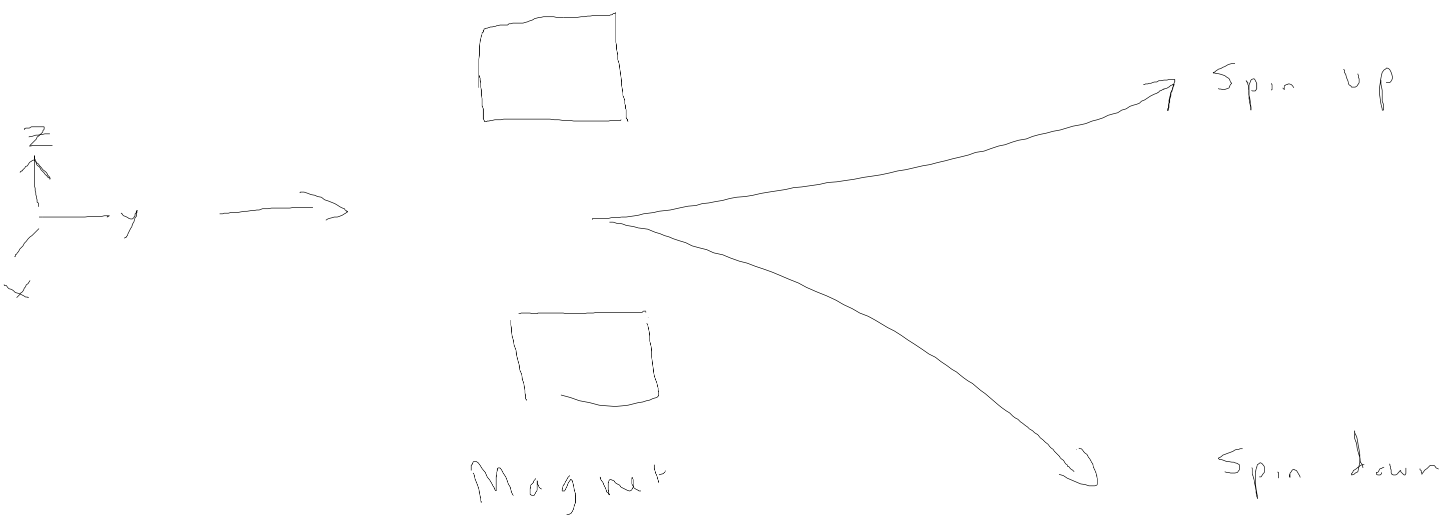
$$H = -\mu \cdot B = -\gamma B \cdot S$$

Stern-Gerlach experiment

Assume you have a non uniform magnetic field

$$\vec{B} = \alpha z \hat{k}$$

Note this violates $\vec{\nabla} \cdot \vec{B} = 0$ Field will also have to vary in x or y direction see text



Classical Force

$$F = -\nabla H = \vec{\nabla}(\vec{\mu} \cdot \vec{B})$$

$$F = \gamma \alpha S_z$$

Spin up $S_z > 0$ bent up, Spin down pushed down.

Classically expect cont. distribution of S_z so that will be some particles with $S_z = 0$ (notice this is not an eigen state!) which will not be deflected.

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In 1926 Stern-Gerlach discovered spin of electron.

Oven of silver atoms. Most electrons in atom pair off to leave only last spin 1/2 electron. We will see how to add spins in a moment but for now say spin up + spin down electrons can cancel to give a spin 0.

Pass beam through inhomogenious magnetic field. Beam split into two components.

Inserted post card after magnet and observed two spots. ⇒ Mailed post card to Niels Bohr.

Note in quantum mech. a measurement of S_z must yield an eigenvalue $s_z = 0$ is not an option!

How do you prepare a state? How do you make a measurement?

In QM these are closely related.

If you want to prepare a spin up state, use a Stern-Gerlach magnet and block off the lower path. All atoms going on upper path have a spin up electron.

If you want to measure the spin of the electron use a Stern Gerlach magnet and detectors which count the # going up and going down.

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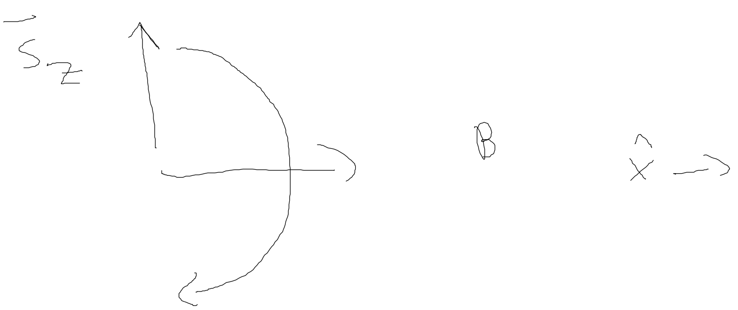
Why does a measurement of the x component of the spin disturb the z component?

To measure the x component need a nonhomogenious B field in x direction. B field in x direction causes $\hat{z}$ component of spin to precess.



direction of spin rotates about B axis