Lecture #28 Y 11/6/98

Angular Momentum

$$\vec{L} = \vec{r} \wedge \vec{p}$$

$$L_x = y p_z - z p_y \qquad L_y = z p_x - x p_z$$

$$L_z = x p_y - y p_x$$

$$L_x = \frac{\hbar}{i}\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}\right) \qquad L_y = \frac{\hbar}{i}\left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}\right)$$

$$L_z = \frac{\hbar}{i}\left(x \frac{\partial}{\partial} - y \frac{\partial}{\partial x}\right)$$

Commutator of $[L_x, L_y] \neq 0$

$$[L_x, L_y] f = L_x L_y f - L_y L_x f$$

$$L_x L_y f = -\hbar^2 \left[ y \frac{\partial}{\partial z}\left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}\right) f - z \frac{\partial}{\partial y}\left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}\right) f \right]$$

$$= -\hbar^2 \, y \frac{\partial}{\partial x} f \quad + 2^{nd} \text{ der. on } f \text{ terms}$$

$$L_y L_x f = -\hbar^2 \left[ z \frac{\partial}{\partial x}\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}\right) f - x \frac{\partial}{\partial z}\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}\right) f \right]$$

$$= -\hbar^2 \, x \frac{\partial}{\partial y} f$$

$$[L_x, L_y] = -\hbar^2 \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}\right) f$$

$$\boxed{[L_x, L_y] = i\hbar L_z}$$

also $[L_y, L_z] = i\hbar L_x$, $[L_z, L_x] = i\hbar L_y$

Thus we can't have a simultanious eigenstate of $L_z$ and $L_x$ etc.

However

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2$$

does commute

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$$

example

$$[L^2, L_x] = \underbrace{[L_x^2, L_x]}_{=0} + [L_y^2, L_x] + [L_z^2, L_x]$$

$$= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x]$$
$$+ [L_z, L_x]L_z$$

$$= L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y)$$
$$+ (i\hbar L_y)L_z$$

$$= 0$$

Used $[A^2, B] = A[A, B] + [A, B]A$

$$= A(AB - BA) + (AB - BA)A = A^2B - BA^2 \checkmark$$

Can write $L^2, L_z$ in spherical coordin'ns

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

Note $-\hbar^2\nabla^2 = \vec{p}^{\,2} = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}L^2$

Thus $L^2$ is angular part of $\vec{p}^{\,2}$

<u>Eigenstates</u> of $L^2, L_z$

$$L_z e^{im\phi} = \hbar m e^{im\phi}$$

This is part of $Y_l^m(\theta,\phi)$. Indeed

$$L^2 Y_l^m(\theta,\phi) = \hbar^2 l(l+1) Y_l^m(\theta,\phi)$$

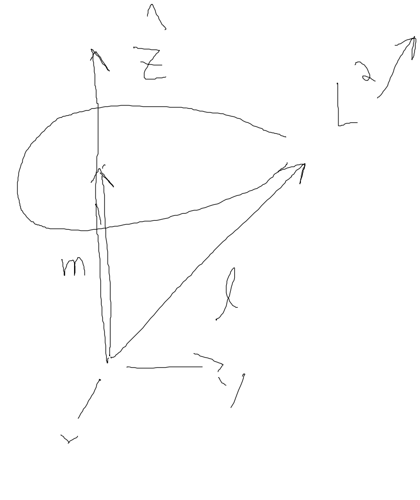
Thus a $Y_l^m$ is simultanious eigenstate of $L^2$ and ~~Bz~~ $L_z$

$$L^2 Y_l^m = \hbar^2 l(l+1) Y_l^m$$

$$L_z Y_l^m = \hbar m Y_l^m$$

With $|m| \leq l$

$m$ is $z$ projection of $L$



## Spin

Can have orbital angular momentum about some origin ($e^-$ about nucleus in atom, Earth about sun)

Can also have angular momentum about an objects center of mass. (Earth spins on its axis)

This intrinsic angular momentum can be described with spin operators

$$S_x, \quad S_y, \quad Z_z \quad \text{and} \quad S^2 = S_x^2 + S_y^2 + S_z^2$$

These have the same commutation relations

$$[L_x, L_y] = i\hbar L_z \Rightarrow [S_x, S_y] = i\hbar S_z$$

$$[S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

there are eigenstates $|s, m\rangle$ so that

$$S^2 |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle$$

and $\quad S_z |s, m\rangle = \hbar m |s, m\rangle$

Howeve no simple picture of $|s, m\rangle$ as a wave function in coordinate space.

Spin is an <u>internal</u> degree of freedom. An electron at a given point in space can either be spin up or spin down

Spin can either be integer (boson) or 1/2 integer (Fermion)

Example $\pi$ meson has spin $s=0$

$|m| \leq s$ so only one state $m=0$

$S^2|00\rangle = 0$ $\quad S_z|00\rangle = 0$

Deuteron is spin one

$s=1$ $\quad m=-1, 0, 1$ could be in one of three states

electron is spin $\frac{1}{2}$ $\quad m=-\frac{1}{2}$ or $+\frac{1}{2}$

Thus an electron is either spin up $\uparrow$ or spin down $\downarrow$

Full wave function

$$|\psi\rangle = \psi_{nlm}(r, \theta, \phi)\ |\tfrac{1}{2}\, m_s\rangle$$

Coordinate space wave function

spin wave function

Note Orbital angular momentum must be integer $l = 0, 1, 2$

Spin can be $\frac{1}{2}$ integer

Can represent spin wave function as a two component vector

$$|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

spin up $\qquad$ spin down

Then $S^2$ and $S_z$ are $2 \times 2$ matricies

$S_x = \frac{\hbar}{2}\sigma_x \qquad S_y = \frac{\hbar}{2}\sigma_y \qquad S_z = \frac{\hbar}{2}\sigma_z$

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The $\sigma$ matiicies are hermitian $\sigma^\dagger = \sigma$

$$[\sigma_x, \sigma_y] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = 2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[\sigma_x, \sigma_y] = 2i\,\sigma_z$$

so $[S_x, S_y] = \frac{\hbar}{2}(2i)S_z = i\hbar S_z$ ✓

$$S_z\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S_z\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$