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Lecture #38 Fine structure of Hydrogen

$$H^0 = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r} \qquad O(\alpha^2 mc^2)$$

$H' =$ Relativistic + Spin orbit $\Big\}$ $\alpha^4 mc^2$

+ Lamb shift $\quad \alpha^5 mc^2$

+ Hyperfine $\quad \alpha^4 \frac{m}{m_p} mc^2$

Relativistic correction

$$H'_r = \underbrace{\sqrt{p^2c^2 + m^2c^4} - mc^2}_{\text{full kinetic energy}} - \frac{p^2}{2m}$$

what is in H^0

$$H'_r \simeq -\frac{p^4}{8m^3c^2}$$

First order correction

$$E'_r = \langle H'_r \rangle = -\frac{1}{8m^3c^2}\langle \psi | p^4 | \psi \rangle$$

$$= -\frac{1}{8m^3c^2}\langle p^2\psi | p^2\psi \rangle$$

$$H^0\psi = E\psi \implies \hat{p}^2\psi = 2m(E-V)\psi$$

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$$E_r^1 = -\frac{1}{2mc^2} \langle (E - V)^2 \rangle$$

$$= -\frac{1}{2mc^2} \left[E^2 - 2E\langle V \rangle + \langle V^2 \rangle \right]$$

$$E = E_n^{(0)} = E_n \qquad V = \frac{e^2}{4\pi\epsilon_0} \frac{1}{r}$$

$$E_r^1 = -\frac{1}{2mc^2} \left[E_n^2 - 2E_n \frac{e^2}{4\pi\epsilon_0} \left\langle \frac{1}{r} \right\rangle + \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \left\langle \frac{1}{r^2} \right\rangle \right]$$

$$\left\langle \frac{1}{r} \right\rangle = \frac{1}{n^2 a}$$

remember $\psi_{nlm} = N_{nl}\, e^{-r/na} \left(\frac{2r}{na}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right) Y_l^m(\theta,\phi)$

$$\left\langle \frac{1}{r^2} \right\rangle = \frac{1}{(l+\frac{1}{2}) n^3 a^2}$$

$$E_n = -\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \frac{1}{n^2}$$

$$a = \frac{4\pi\epsilon_0 \hbar^2}{me^2}$$

$$E_r^1 = -\frac{E_n^2}{2mc^2} \left[\frac{4n}{l+\frac{1}{2}} - 3 \right]$$

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Correction of order $\frac{E_n}{mc^2} \sim 2 \times 10^{-5}$

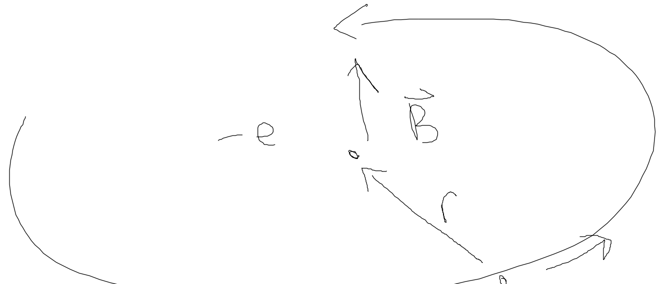
Spin-Orbit Coupling

Also of order $\frac{E_n}{mc^2}$ $\quad E_n = \alpha^4 mc^2$

In rest frame of electron, moving proton gives rise to a magnetic field

$$\vec{B} = \frac{1}{4\pi\epsilon_0} \frac{e}{mc^2 r^3} \vec{L}$$

Dipole magnetic field in direction $\vec{L}$



electron at rest

Current loop of proton

$B = \frac{\mu_0}{2r} I$ (Biot-Savart Law) $\quad I = \frac{+e}{T}$

T = Period of orbit

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$$L = r\,mv = 2\pi m r^2 / T$$

so

$$B = \frac{\mu_0}{2r} \frac{e\,L}{2\pi m r^2}$$

$$c^2 = 1/\mu_0 \epsilon_0 \Rightarrow \mu_0 = \frac{1}{\epsilon_0 c^2}$$

$$B = \frac{e}{4\pi\epsilon_0 c^2} \frac{L}{m r^3}$$

Spin orbit Hamiltonian

$$H = -\vec{\mu}_e \cdot \vec{B}$$

Magnetic dipole moment of electron.

Classical charge distribution with spin S and charge q

$$\mu = \frac{q}{2m} S$$

From relativistic effects, Dirac equation result is a factor of g=2 bigger and q = -e

$$\vec{\mu}_e = -\frac{g e}{2m} S = -\frac{e}{m} \vec{S}$$

$$H = \frac{e^2}{4\pi\epsilon_0} \frac{1}{m^2c^2r^3} \vec{S}\cdot\vec{L}$$

but rest frame of electron is a noninertial frame. Correction called <u>Thomas</u> <u>precession</u> reduces result by factor of two

$$\boxed{H'_{so} = \frac{e^2}{8\pi\epsilon_0} \frac{1}{m^2c^2r^3} \vec{S}\cdot\vec{L}}$$

$$\vec{J} = \vec{L} + \vec{S}$$

$$\vec{J}^2 = L^2 + S^2 + 2L\cdot S$$

$$J^2\psi_{nlm} = \hbar^2\left[l(l+1) + \tfrac{1}{2}\tfrac{3}{2}\right] + 2L\cdot S\psi_{nlm}$$

$$= \hbar^2 j(j+1)\psi_{nlm}$$

j = total angular momentum quantum #

$= l + \frac{1}{2}$ or $l - \frac{1}{2}$

$$L\cdot S\,\psi_{nlm} = \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]$$

with $s = 1/2$

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{l\,(l+\frac{1}{2})\,(l+1)\,n^3 a^3}$$

$$E'_{so} = \langle H'_{so} \rangle = \frac{e^2}{8\pi\epsilon_0}\;\frac{1}{m^2c^2}\;\frac{\hbar^2/2\,\left[j(j+1) - l(l+1) - 3/4\right]}{l(l+\frac{1}{2})(l+1)\,n^3 a^3}$$

$$E'_{so} = \frac{E_n^2}{mc^2}\;n\left[\frac{j(j+1) - l(l+1) - 3/4}{l\,(l+1)\,(l+\frac{1}{2})}\right]$$

$$E'_{fs} = E'_{so} + E'_{r} = \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right)$$

$$E_{nj} = -\frac{13.6\text{ eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$