3/30/01

Lecture 29 Electron in a Magnetic Field

Raising + Lowering op. ($L_\pm = L_x \pm iL_y$)

$$S_\pm = S_x \pm i S_y$$

$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pm i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad S_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$S_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad S_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

Note in prob. 4.19 showed

$$L_+ f_\ell^m = A_\ell^m f_\ell^{m+1}$$

Consider $S_+ f_{1/2}^{-1/2} = A_{1/2}^{-1/2} f_{1/2}^{1/2}$

showed

$$A_\ell^m = \hbar \sqrt{\ell(\ell+1) - m(m+1)}$$

$$A_{1/2}^{-1/2} = \hbar \sqrt{\frac{3}{4} - \frac{-1}{2}\left(-\frac{1}{2}+1\right)} = \hbar \checkmark$$

Note Prob. Set #9 Due Monday Apr 2 not Wed.

What are eigenstates of S_x?

$$S_x \chi = \lambda \chi$$

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det \begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0 = \lambda^2 - \hbar^2/4$$

$\lambda = \pm \hbar/2$ same as for S_z

can be either spin up $+\hbar/2$ along x or spin down $-\hbar/2$ along x

Eigenvectors $\lambda = +\hbar/2$

$$\left(-\hbar/2 \, , \; \hbar/2\right)\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$\Rightarrow a = +b$ normalize

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$S_x \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$S_x \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Spin up along z $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, spin up along x $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Can expand any state in a complete set of eigenstates. Start with spin up along x

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\chi_+^{(z)} + \frac{1}{\sqrt{2}}\chi_-^{(z)}$$

with $\chi_+^{(z)} = \begin{pmatrix}1\\0\end{pmatrix}$ $\chi_-^{(z)} = \begin{pmatrix}0\\1\end{pmatrix}$

Thus if state is spin up along x it is a 50/50 mixture of spin up and spin down along z

(1) Prepare system to be spin up along $\hat{x}$

(2) Measure S_z

$c_+ = \frac{1}{\sqrt{2}}$ 50% chance spin up

$c_- = \frac{1}{\sqrt{2}}$. $|c_-|^2 = \frac{1}{2}$ 50% spin down.

Of course if prepare system to be spin up along $\hat{x}$ and measure S_x get spin up with 100% prob.

(1) Or prepare system to be spin up along $\hat{z}$

$$\chi = \begin{pmatrix}1\\0\end{pmatrix}$$

(2) Measure S_x

Expand χ in S_x eigenstates

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\chi = \sum_{m_s=-\frac{1}{2}}^{\frac{1}{2}} \chi^{(x)}_{m_s} c_{m_s}$$

$$c_{m_s} = \chi^{(x)\dagger}_{m_s} \chi \Rightarrow$$

$$c_{+\frac{1}{2}} = \frac{1}{\sqrt{2}}(1,1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$c_{-\frac{1}{2}} = \frac{1}{\sqrt{2}}(1,-1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$$

50 % prob. to get spin up
50 % prob. to get spin down.

You can't be in a simultanious eigenstate of both S_x and S_z because

$$[S_x, S_z] \neq 0$$

Can always expand given inital state in eigenstates of any operator S_z, S_x, S_y, ...

Expansion coef squared give prob. of measurements.

(1) Measure spin up along $\hat{z}$
(2) After measurement $\chi = \begin{pmatrix}1\\0\end{pmatrix}$
(3) Measure S_x
(i) get spin up with 50% chance
(4) After a measurement which gave spin up
$$\chi = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$
(5) Measure S_z again
50 % chance to get spin down.

Election in a Magnetic Field

Spinning charged particle is a magnetic dipole

$$\vec{\mu} = \gamma \vec{S}$$

γ = gyromagnetic ratio

$\gamma = \frac{q}{2m}$ classically $\quad q$ = charge

$\gamma = g\left(\frac{q}{2m}\right)$ in relativistic quantum mechanics

$g \approx 2$ from relativistic and Quantum fluctuation effects.

Energy of magnetic dipole in B field

$$H = -\vec{\mu} \cdot \vec{B}$$
$$= -\gamma \vec{B} \cdot \vec{S}$$

Chose $\hat{z}$ to be along direction of $\vec{B}$

$$\vec{B} = B_0 \hat{z}$$

$$H = -\gamma B_0 S_z = -\left(\frac{\gamma B_0 \hbar}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$H \chi_+ = E_+ \chi_+ \qquad E_+ = -\gamma B_0 \hbar/2$

$H \chi_- = H\begin{pmatrix} 0 \\ 1 \end{pmatrix} = E_- \chi_- \qquad E_- = +\gamma B_0 \hbar/2$

$$i\hbar \frac{\partial \chi}{\partial t} = H\chi$$

let $\chi(0) = \begin{pmatrix} a \\ b \end{pmatrix}$ $\qquad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ etc.

$$\chi(t) = a\,\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar}$$

$$= \begin{pmatrix} a e^{i\gamma B_0 t/\hbar} \\ b e^{-i\gamma B_0 t/\hbar} \end{pmatrix}$$

Since $|a|^2 + |b|^2 = 1$

let $a = \cos\alpha/2 \qquad b = \sin\alpha/2$

$$\chi(t) = \begin{bmatrix} \cos\alpha/2 \; e^{i\gamma B_0 t/2} \\ \sin\alpha/2 \; e^{-i\gamma B_0 t/2} \end{bmatrix}$$

calculate

$$\langle S_x \rangle = \chi(t)^\dagger \, S_x \, \chi(t)$$

$$= \frac{\hbar}{2}\left(\cos\alpha/2 \; e^{-i\gamma B_0 t/2},\ \sin\alpha/2 \; e^{i\gamma B_0 t/2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} \cos\alpha/2 \; e^{i\gamma B_0 t/2} \\ \sin\alpha/2 \; e^{-i\gamma B_0 t/2} \end{bmatrix}$$

$$= \frac{\hbar}{2}\cos\alpha/2 \, \sin\alpha/2 \left[e^{-i\gamma B_0 t} + e^{i\gamma B_0 t}\right]$$

$$= \frac{\hbar}{2}\sin\alpha \, \cos\gamma B_0 t$$

$$\left[2\cos\alpha/2 \, \sin\alpha/2 = \sin\alpha\right]$$

Likewise

$$\langle S_y \rangle = \chi^\dagger S_y \chi = -\frac{\hbar}{2}\sin\alpha \, \sin(\gamma B_0 t)$$

$$\langle S_z \rangle = \chi^\dagger S_z \chi = \frac{\hbar}{2}\left(\cos^2\alpha/2 - \sin^2\alpha/2\right) = \frac{\hbar}{2}\cos\alpha$$