3/28/01

Lecture 28 Angular Momentum Cont.

$[L^2, L_i] = 0$

$L_\pm \equiv L_x \pm i L_y$

$[L_z, L_\pm] = \pm \hbar L_\pm$

from $[L_z, L_x] = i\hbar L_y$ etc.

Since $[L^2, L_z] = 0$ can diagonalize both L^2 and L_z

$L^2 f = \lambda f \qquad L_z f = \mu f$

Define $g = L_+ f$

$L^2 g = L^2 L_+ f = L_+ L^2 f = \lambda L_+ f$
$= \lambda g$

Thus g is also an eigenstate of L^2 with same eigenvalue

$L_z g = L_z L_+ f = [L_z, L_+] f + L_+ L_z f$
$= \hbar L_+ f + \mu L_+ f \quad , \qquad L_z f = \mu f$
$= (\mu + \hbar) L_+ f = (\mu + \hbar) g$

Thus g is an eigenstate of L_z with new eigenvalue $\mu + \hbar$

L_+ raises z component of ang. momentum by $\hbar$.

Now try $L_+ g$

$L_z L_+ g = [L_z, L_+] g + L_+ L_z g$
$= (\mu + 2\hbar) g$

Thus $L_+ g$ is an eigenstate of L_z with $\mu + 2\hbar$

Cant keep going forever. Otherwise L_z component would exceed total ang. mom

$$\langle L^2 \rangle \geq \langle L_z^2 \rangle$$

since $\langle L^2 \rangle = \langle L_z^2 \rangle + \langle L_x^2 + L_y^2 \rangle$

There must be a maximum value so that

$$L_z f_l = \hbar l \, f_l$$

and $L_+ f_l = 0$

Try to raise z component by one and get nowhere.

$$L_z f_l = \hbar l \, f_l$$
$$L^2 f_l = \lambda f_l$$

Now $(L_x + iL_y)(L_x - iL_y) = L_+ L_-$

$= L_x^2 + L_y^2 + i(L_y L_x - L_x L_y) = L_x^2 + L_y^2 + \hbar L_z$

so $L^2 = L_x^2 + L_y^2 + L_z^2 = L_+ L_- + L_z^2 - \hbar L_z$

or $\boxed{L^2 = L_- L_+ + L_z^2 + \hbar L_z}$

$$L^2 f_l = L_- L_+ f_l + \left[(\hbar l)^2 + \hbar \hbar l \right] f_l$$

but $L_+ f_l = 0$ so

$$L^2 f_l = \hbar^2 l(l+1) f_l = \lambda f_l$$

Note

$$L_- f_l = \hbar(l-1) f_l$$

L_+ is a raising operator and L_- is a lowering operator

Angular momentum spectrum

L_z has eigenvalues from $-\hbar l$ to $\hbar l$ in steps of $\hbar$

$$L_- f_l^{-l} = 0$$ Lowering the lowest state

and $L^2 f_l = \hbar^2 l(l+1) f_l$

clearly $f = Y_l^m(\theta, \phi)$

from before. Found spectrum

$$-l \leq m \leq l$$

and $\lambda = \hbar^2 l(l+1)$

just from commutation relations

Note states of different m values differ by one unit.

For orbital angular momentum m, l are integers

$$Y_l^m(\theta, \phi + 2\pi) = Y_l^m(\theta, \phi)$$

because ϕ dep. was $e^{im\phi}$ and $e^{i2\pi} = 1$

Nature also makes a bizarre choice
$l, m = \frac{1}{2}$ integer

$$\psi(\phi + 2\pi) = -\psi(\phi)$$

Double valued. But

$$\psi^* \hat{O} \psi$$

is single valued

$$\psi^*(\phi + 2\pi)\, \hat{O}\, \psi(\phi + 2\pi) = \psi^*(\phi)\, \hat{O}\, \psi(\phi)$$

Spin angular momentum

Abstract spin states

$$S^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$$

$$S_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle$$

s, m_s can be integer bosons
or $\frac{1}{2}$ integer fermions

Define abstract spin operators by commutation relations

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x$$

$$[S_z, S_x] = i\hbar S_y$$

$$[S^2, S_i] = 0 \qquad i = x, y, z$$

Full H atom wavefunction includes spatial and spin parts

$$|\psi_{nlm m_s}\rangle = R_{nl}(r)\, Y_l^m(\theta, \phi)\, |\tfrac{1}{2}\ m_s\rangle$$

For $s = \frac{1}{2}$ two states $m_s = +\frac{1}{2}, -\frac{1}{2}$

Let $|\frac{1}{2}\ \frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |\frac{1}{2}\ -\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

General state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$

$\chi^\dagger = (a^*, b^*)$

Norm $\chi^\dagger \chi = 1 = a^* a + b^* b$

Operators

$S^2 \chi = \hbar^2 \frac{1}{2}(\frac{1}{2}+1)\chi$

$\Rightarrow S^2 = \frac{3}{4}\hbar \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$S_z \chi = \frac{\hbar}{2}\begin{pmatrix} a \\ -b \end{pmatrix}$ $\quad$ & $\quad S_z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$S_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Chose S_x, S_y so that

$S_x^\dagger = S_x$

and $[S_x, S_y] = i\hbar S_z$

$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$S_x S_y = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$$S_y S_x = \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$[S_x, S_y] = S_x S_y - S_y S_x = i \frac{\hbar^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= i\hbar \; \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar \, S_z$$