Lecture 25  Angular Momentum

\[
\begin{align*}
L &= r \times p \\
L_x &= \frac{\hbar}{i} (y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}) \\
\left[ L_X, L_Y \right] &= i \hbar L_Z \\
\left[ L_Z, L_X \right] &= i \hbar L_Y \\
L_x^2 &= L_x^2 + L_y^2 + L_z^2 \\
\left[ L_x^2, L_x \right] &= \left[ L_y^2, L_x \right] + \left[ L_z^2, L_x \right] \\
&= L_y^2 \{ L_x, h_x \} + L_z^2 \{ L_x, h_z \} + L_x \{ L_z^2, h_z \} + L_z \{ L_z^2, h_x \} \\
&= i \hbar L_y \{-L_z\} + i \hbar \{L_z, -L_z\} + L_z \{i \hbar L_y + i \hbar L_x\} \\
&= 0 = \left[ L^2, L_y \right] = \left[ L^2, L_z \right]
\end{align*}
\]

Can simultaneously diagonalize \( L^2 \) and one of the \( L_i \)'s choose \( L_z \)

\[
L^2 f = \lambda f \quad \text{and} \quad L_z f = \mu f
\]

Define \( L_\pm = L_x \pm i L_y \)

\[
\left[ L_z, L_\pm \right] = \left[ L_z, L_x \right] \pm i \left[ L_z, L_y \right] = \pm i \hbar L_\pm
\]

\[
g = L_\pm f
\]

\[
L^2 g = L^2 L_\pm f = L_\pm L^2 f = \lambda L_\pm f = \lambda g
\]

Note \( \left[ L^2, L_\pm \right] = 0 \) since \( \left[ L^2, L_x \right] = \left[ L^2, L_y \right] = 0 \)
Thus $g$ is still an eigenstate of $L^2$ with the same eigenvalue.

\[ L^2 g = L^2 (L^+ f) = \{ L^2, L^+ \} f + hL^2 f \]
\[ = \pm h L^+ f + \mu L^2 f \]
\[ = (\mu \pm \hbar) L^+ f = (\mu \pm \hbar) g \]

Thus $g$ is an eigenstate of $L^2$ with a new eigenvalue raised or lowered by $\hbar$.

$L^+$ is a raising operator.
$L^-$ is a lowering operator.

Can't raise forever, eventually $L$ compares.

\[ \langle L^2 \rangle \geq \langle L z \rangle \]

Thus there must be a maximum value.

\[ L^2 f = \pm L^+ f \]

When $L^+ f = 0$.

Note $L^2 f = L f L f$.

\[ L^2 = L^+ L^- + L^2 - \hbar L_z \]
\[ = (L_x + \imath L_y)(L_x - \imath L_y) + L^2 - \hbar L_z \]
\[ = L_x^2 + L_y^2 + L_z^2 + \imath [L_y L_x + L_x L_y] - \hbar L_z \]
\[ = L_x^2 + L_y^2 + L_z^2 - \hbar L_z \]

Also

\[ L^2 = L^- L^+ + L^2 + \hbar L_z \]
\[ L^2 F_b = L^2 f_b + (2 L^2 f_b + (2 L^2 f_b + \hbar L f_b = 0 + \hbar^2 l(l+1) f_b \]

So \[ \lambda = \hbar^2 l(l+1) \]

The eigenvalue of the square of the angular momentum is \( \hbar^2 l(l+1) \), where \( l \) is the maximum \( z \) projection.

Note that \( L^2 \geq L_z^2 \) and the extra \( \hbar^2 l \) amount of accounts for the minimum \( x \) and \( y \) directions consistent with HUP.

Likewise there is a lowest \( z \) projector

\[ L_z F_b = 0 \]

Assume \( L_z F_b = \hbar \overline{L} F_b \) and \( L^2 F_b = \hbar^2 (L(L+1)) F_b \)

\[ \overline{L} F_b = (L+L^2 - \hbar^2 L^2) F_b = 0 + (\hbar^2 l^2 - \hbar^2 l) \]

\[ \overline{L} F_b = \hbar^2 l(l+1) F_b \]

Solution \( \overline{L} = -l \)

So allowed \( z \) projections run from \( \hbar l \), \( \hbar (l-1) \), \( \ldots \), \( -\hbar l \)

We have found the spectrum of \( L_z \) just from the commutation relations.
Clearly,
\[ f = Y^m_l(\theta, \phi) \]
\[ L_\perp Y^m_l = \pm i \ell (\ell + 1) Y^m_l \]
\[ L_z Y^m_l = \pm m Y^m_l \]

For orbital angular momenta \( m \) is integer so
\[ f = e^{im\phi} \]

is single valued
\[ Y^m_l(\theta, \phi + 2\pi) = Y^m_l(\theta, \phi) \]

because \( e^{im\phi} = 1 \) for \( m \) integer.

There is another possibility for abstract angular momenta \( m \) can be \( 1/2 \) integer??

This seems bizarre, the wave function is not single valued.
\[ f(\phi) = -f(\phi + 2\pi) \]

because \( e^{i\pi} = -1 \)

However, all physics involves expectation values
\[ \langle \hat{\mathcal{O}} \rangle = \langle \Psi | \hat{\mathcal{O}} | \Psi \rangle \]

and this is single valued since \( \Psi(\phi + 2\pi) = \Psi(\phi) \) and \( \Psi(\phi + 2\pi) = \Psi(\phi) \)
Spin angular momentum

Possible to have abstract spin

\[ S^2 |s, m_s\rangle = \hbar^2 s (s+1) |s, m_s\rangle \]

\[ S^2 |s, m_s\rangle = \hbar m_s |s, m_s\rangle \]

with \( s, m_s \) \( \frac{1}{2} \) integer

Example \( s = \frac{1}{2} \)

\[ m_s = \frac{s+1}{2} \quad \frac{1}{2} \]

two possible

Define abstract spin operators by commutation relations

\[ [S_x, S_y] = i \hbar S_z \]

\[ [S_z, S_x] = i \hbar S_y \]

\[ [S^2, S_i] = 0 \]

Spin is an internal degree of freedom point electron carries spin fixed at one point in space

No spatial dependence to the spin wave function

Full electron wave function in \( H \) atom

\[ |\psi_{nm\ell m_s}\rangle = R_{\ell n}(r) Y^m_\ell (\hat{\theta}, \hat{\phi}) |\frac{1}{2} m_s\rangle \]

Electron is spin \( \frac{1}{2} \) have extra quantum \( \ell + \frac{1}{2} \) \( \ell - \frac{1}{2} \) to fully describe state.