3/22/00

Lecture 25 Angular Momentum

$L = r \wedge p$

$L_x = \frac{\hbar}{i}\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}\right)$ etc.

$[L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x$

$[L_z, L_x] = i\hbar L_y$

$L^2 = L_x^2 + L_y^2 + L_z^2$

$[L^2, L_x] = [L_y^2, L_x] + [L_z^2, L_x]$

$= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z$

$= i\hbar L_y(-L_z) + i\hbar(-L_z L_y) + L_z(i\hbar L_y) + i\hbar L_y L_z$

$= 0 = [L^2, L_y] = [L^2, L_z]$

Can simultaniously diagonalize L^2 and one of the L_i chose L_z

$L^2 f = \lambda f$ and $L_z f = \mu f$

Define $L_\pm = L_x \pm i L_y$

$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y]$

$= \pm \hbar L_\pm$

$g = L_\pm f$

$L^2 g = L^2 L_\pm f = L_\pm L^2 f = \lambda L_\pm f = \lambda g$

note $[L^2, L_\pm] = 0$ since $[L^2, L_x] = [L^2, L_y] = 0$

Thus g is still an eigenstate of L^2 with the same eigenvalue

$$L_z g = L_z L_\pm f = [L_z, L_\pm] f + L_\pm L_z f$$

$$= \pm \hbar L_\pm f + \mu L_\pm f$$

$$= (\mu \pm \hbar) L_\pm f = (\mu \pm \hbar) g$$

Thus g is an eigenstate of L_z with a new eigenvalue raised or lowered by $\hbar$

L_+ is a raising operator
L_- is a lowering operator

Can't raise forever eventually z component will exceed total

$$\langle L^2 \rangle \geq \langle L_x^2 \rangle$$

Thus there must be a maximum value

$$L_z f_l = \hbar l f_l$$

where $L_+ f_l = 0$

note $L^2 f_l = \lambda f_l$

$$L^2 = L_+ L_- + L_z^2 - \hbar L_z$$

$$= (L_x + i L_y)(L_x - i L_y) + L_z^2 - \hbar L_z$$

$$= L_x^2 + L_y^2 + L_z^2 + i[L_y L_x - L_x L_y) - \hbar L_z$$

$$= L_x^2 + L_y^2 + L_z^2 \quad (i(-i\hbar L_z) - \hbar L_z) \checkmark$$

also $L^2 = L_- L_+ + L_z^2 + \hbar L_z$

$$L^2 f_l = L_- L_+ f_l + L_z^2 f_l + \hbar L_z f_l$$
$$= 0 + (\hbar l)^2 f_l + \hbar^2 l f_l$$
$$= \hbar^2 l(l+1) f_l$$

So $\boxed{\lambda = \hbar^2 l(l+1)}$

The eigenvalue of the square of the angular momentum is $\hbar^2 l(l+1)$ where $\hbar l$ is the maximum z projection

Note $L^2 \geq L_z^2$ and the extra $\hbar^2 l$ over $\hbar^2 l^2$ accounts for the minimum amount of angular momentum in the x and y directions consistent with HUP

Likewise there is a lowest z projection

$$L_- f_b = 0$$

assume $L_z f_b = \hbar \bar{l} f_b$ and $L^2 f_b = \hbar^2 l(l+1) f_b$

$$L^2 f_b = (L_+ L_- + L_z^2 - \hbar L_z) f_b$$
$$= 0 + (\hbar^2 \bar{l}^2 - \hbar^2 \bar{l}) f_b = \hbar^2 l(l+1) f_b$$

solution $\bar{l} = -l$

So allowed z projections run from

$$\hbar l, \hbar(l-1), \ldots, -\hbar l$$

We have found the spectrum of L_z just from the commutation relations.

Clearly

$$f = Y_l^m(\theta, \phi)$$

$$L^2 Y_l^m = \hbar^2 l(l+1) Y_l^m$$

$$L_z Y_l^m = \hbar l Y_l^m$$

For orbital angular momenta m is integer so

$$f = e^{im\phi}$$

is single valued

$$Y_l^m(\theta, \phi + 2\pi) = Y_l^m(\theta, \phi)$$

because $e^{i2\pi m} \equiv 1$ for m integer.

There is another possibility for abstract angular momenta

m can be $1/2$ integer ??

This seems bezar. The wave function is not single valued

$$f(\phi) = -f(\phi + 2\pi)$$

because $e^{i\pi} = -1$

Howevere all physics involves expectation values

$$\psi^* \hat{O} \psi$$

and this is single valued since $\psi^*(\phi + 2\pi) = -\psi^*(\phi)$ and $\psi(\phi + 2\pi) = -\psi(\phi)$

Spin angular momentum

Possible to have abstract spin states

$$S^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$$

$$S_z |s, m_s\rangle = \hbar m |s, m_s\rangle$$

with s, m_s ½ integer

Example $s = \frac{1}{2}$ $m_s = \begin{cases} +\frac{1}{2} \\ -\frac{1}{2} \end{cases}$ two possibilities

Define abstract spin operators by commutation relations

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x$$

$$[S_z, S_x] = i\hbar S_y$$

$$[S^2, S_i] = 0$$

Spin is an internal degree of freedom. Point electron carries spin angular momentum even if it is fixed at one point in space.

No spatial dependence to the spin wave function

Full electron wave function in H atom

$$|\psi_{n l m m_s}\rangle = R_{nl}(r)\, Y_l^m(\theta, \phi)\, |\tfrac{1}{2}\, m_s\rangle$$

Electron is spin ½ have extra quantum # $+\frac{1}{2}$ or $-\frac{1}{2}$ to fully describe state.