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Lecture 26 Review

Postulates of QM

1.) The state of a particle is represented by a normalized vector ($|\Psi\rangle$) in the Hilbert space L_2

(L_2 set of all square integrable functions)

2.) Observable quantities $Q(x, p, t)$ are represented by Hermitian operators

$$\hat{Q}\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}, t\right);$$

the expectation value of Q in the state $|\Psi\rangle$ is $\langle \Psi | \hat{Q} \Psi \rangle$

Take classical quantity $Q(x, p, t)$ and replace $x \to \hat{x}$ and p with $\hat{p}$ to get $\hat{Q}$. Simple case $\hat{x} = x$

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

3.) A measurement of the observable Q in the state $|\Psi\rangle$ is <u>certain</u> to yield λ iff $|\Psi\rangle$ is an eigen state of $\hat{Q}$ with eigen value λ

Example $\hat{H}\psi = E\psi$

3'.) If you measure an observable Q on a particle in state $|\Psi\rangle$ you are certain to get one of the eigenvalues of $\hat{Q}$. The probability of getting λ is the absolute square of the λ component of $|\Psi\rangle$ $P_\lambda = |\langle \lambda | \Psi \rangle|^2$

Eigenvectors are complete. Can expand any function

If eigenvectors are discreate

$$|\Psi\rangle = \sum_{n=1}^{\infty} c_n |e_n\rangle$$

with

$$\hat{Q}|e_n\rangle = \lambda_n |e_n\rangle$$

$$\langle e_n|e_m\rangle = \delta_{nm}$$

Eigenvectors are orthonormal

Project to find c_n

$$\langle e_n|\Psi\rangle = c_n$$

$$P_n = |c_n|^2 = |\langle e_n|\Psi\rangle|^2$$

If eigenvectors are cont.

$$\langle e_k|e_l\rangle = \delta(k-l)$$ Dirac delta func.

$$|\Psi\rangle = \int_{-\infty}^{\infty} dk \; c_k |e_k\rangle$$

$$c_k = \langle e_k|\Psi\rangle$$

$$P_k = |c_k|^2 \, dk = \underbrace{|\langle e_k|\Psi\rangle|^2}_{\text{Prob. density}} dk$$

$P_{k<k+dk}$ is prob. to have k between k and

Generalized Uncertainty Principle

$$\sigma_A^2 \, \sigma_B^2 \geq \left| \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right|^2$$

σ_A^2 = Variance = $\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$

σ_A = Standard deviation = $\left[\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \right]^{1/2}$

$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ commutator

(1) Any single observable can be known to arbitrary precision. As can any number of mutally commuting observables

(2) If two operators do not commute the corresponding observables can't both be known to arbitrary precision

Time dependence of Expectation Values

Time dependence from time dep. of Wave function

$$i\hbar \frac{\partial |\Psi\rangle}{\partial t} = \hat{H} |\Psi\rangle \quad \text{time dep. S. eq.}$$

and time dep. of operator

For stationary states

$$\hat{H} \psi = E \psi$$

so $\Psi(x,t) = e^{-iEt/\hbar} \psi(x)$

In general

$$\frac{d}{dt} \langle \hat{Q} \rangle = \frac{i}{\hbar} \langle [H, \hat{Q}] \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$$

One Dimensional Problems

① Infinite square well $\quad V = \begin{cases} 0 & 0<x<a \\ \infty & \text{else} \end{cases}$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + 0 = E\psi$$

$$\psi = A\sin kx + B\cos kx$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

B.C. $\quad \psi(0) = \psi(a) = 0 \quad \Rightarrow \quad B = 0$

$$\sin ka = 0 \qquad k = \frac{n\pi}{a}$$

$$E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2 \qquad \psi = \sqrt{\frac{2}{a}}\sin(k_n x)$$

Normalize $\quad \int_0^a \psi^*\psi\, dx = 1$

② Harmonic Oscillator $\quad V = \frac{1}{2}kx^2$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{k}{2}x^2\psi = E\psi$$

go to dimensionless variables $\quad z = \sqrt{\frac{m\omega}{\hbar}}\,x$

$\omega = \sqrt{k/m}$ classical frequency

$$\frac{d^2\psi}{dz^2} = (z^2 - K)\psi$$

$$K = E/(\tfrac{1}{2}\hbar\omega)$$

Pull out large z behavior $e^{-z^2/2}$ and expand rest in a power series

Require power series to terminate

so wave function is normalizable

$$E_n = (n+\tfrac{1}{2})\hbar\omega$$

$$\psi_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(z)\, e^{-z^2/2}$$

$H_n(x)$ = Hermite polynomials

$H_0 = 1$ $\quad H_2(z) = 4z^2 - 2$

$H_1 = 2z$ $\quad H_3(z) = 8z^3 - 12z \quad \ldots$

Delta Function pot.

$$V(x) = -V_0\, \delta(x-a)$$

B.C. ① wave function is allways cont.

② derivative is cont. except where pot. is infinite

Integrate S. eq. over small range including a to find discont. in $\frac{d\psi}{dx}$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\,\delta(x-a)\,\psi(x) = E\,\psi(x)$$

$$-\frac{\hbar^2}{2m}\int_{a-\epsilon}^{a+\epsilon} dx\, \frac{d^2\psi}{dx^2} - V_0 \int dx\, \delta(x-a)\,\psi = 0$$

$$\boxed{-\frac{\hbar^2}{2m}\left[\left.\frac{d\psi}{dx}\right|_{a+\epsilon} - \left.\frac{d\psi}{dx}\right|_{a-\epsilon}\right] - V_0\,\psi(a) = 0}$$

Finite square well

Inside if $E - V > 0$
then

$$\psi = A\sin kx + B\cos kx$$

Outside if $E < 0$ (bound state)

$$\psi = Ce^{-\alpha x} + De^{\alpha x}$$

(1) Match ψ and $\frac{d\psi}{dx}$
at edge of well

(2) adjust E so that ψ is
normalizable $\psi \to 0$ as $|x| \to \infty$

Three dimensions

For problems with symmetry use
seperation of variables to seperate
problem into a series of one
Dimensional ones.