Lec. 22  Radial Equation

\[ - \frac{\hbar^2}{2m} \nabla^2 Y(\rho, \phi) + V(\rho) Y(\rho, \phi) = E Y(\rho, \phi) \]

Angular eq.

\[ \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{\partial^2 Y}{\partial \phi^2} = -l(l+1) \sin^2 \theta Y \]

\( l(l+1) \) = Separation constant, indep. of \( \rho, \phi, \theta \)

\( Y(\rho, \phi) = A \frac{p^m}{r} (\cos \theta) e^{im\phi} \)

\[ p_l^m(x) = (1 - x^2)^{l/2} \frac{(d^m}{dx^m}) [P_l(x)] \]

\[ P_l(x) = \frac{1 - x^2}{2^l l! [(d/dx)^l (x^2 - 1)^l}] \]

Note

\[ p_l^m \propto (\frac{d}{dx})^l (x^2 - 1)^l \]

Thus:

\[ p_{lm} = 0 \] for \( m > l \)

\[ \Rightarrow 2l+1 \] possible \( m \) values between \( -l, \ldots, l \)

\[ -l \leq m \leq l \]

Normalize

\[ \int_0^\infty \frac{d^3r}{4\pi} = \int_0^\infty r^2 \sin \theta d\theta d\phi d\rho = 1 \]

\[ \int_0^\infty \frac{d\rho}{\rho} \frac{1}{r} r^2 d\theta d\phi = 1 \]

\[ \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^\pi \sin \theta d\theta \int_0^\infty r^2 dr \frac{|Y|^2}{4\pi} = 1 \]

\[ Y_{lm}(\rho, \phi) = \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} e^{im\phi} \frac{p_l^m}{r} (\cos \theta) \]
\[ k = \text{phase} = \left\{ \begin{array}{ll} \frac{m}{\ell} & m \geq 0 \\ \frac{1}{m} & m < 0 \end{array} \right. \]

Spherical Harmonics are orthogonal
\[ \int_0^{2\pi} \int_0^\pi \sin \theta \, d\theta \, d\phi \, y_\ell^m(\cos \theta, \phi) y_{\ell'}^{m'} = \delta_{\ell \ell'} \delta_{m m'} \]

\( \ell = 0, 1, 2, \ldots \)
\(-\ell \leq m \leq \ell \)

**Radial Equation**
\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} R \right) + \frac{l(l+1)}{2m^2} \frac{1}{r} R + V(r) R = \lambda R \]

Separation constant

Let \( U = r \frac{d}{dr} R \)
\[ R = \frac{R}{r} \]
\[ R' = \frac{U}{r} - \frac{U}{r^2} \]

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} U \right) = \frac{U''}{r} \]
\[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} U + \left[ V(r) + \frac{l(l+1)}{2mr^2} \right] U = \lambda U \]

\[ \Psi(r, \theta, \phi) = \frac{U(r)}{r} y_\ell^m(\theta, \phi) \]

\[ V_{\text{eff}} = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2} \]

\[ \int_0^\infty \rho(r) \, dr = 1 \quad \text{(Centrifugal term)} \]
Only one angular wave function for all central pot.

Example: \( Y_\ell^m(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} \, d\phi \frac{1}{\sqrt{4\pi}} \frac{1}{\sqrt{4\pi}} = 1 \)

Need to specify radial form of pot.

Example: infinite spherical well
\( V(r) = \begin{cases} 0 & r \leq a \\ \infty & r > a \end{cases} \)
\( \psi_{nr} = 0 \quad r > a \)

Inside \( V=0 \)
\( \frac{d^2 \psi}{dr^2} = \left( \frac{\ell(\ell+1)}{r^2} - k^2 \right) \psi \)
\( k = \sqrt{2mE} \alpha \)

Solutions: spherical Bessel and spherical Neumann functions
\( \psi(r) = A_J \, j_\ell(kr) + B_N \, n_\ell(kr) \)

\( j_\ell(x) = (-1)^\ell \frac{1}{\sqrt{\pi} x^\ell} \left( \frac{\sin x}{x} \right) \)
\( n_\ell(x) = -(-1)^\ell \frac{1}{\sqrt{\pi} x^\ell} \left( \frac{\cos x}{x} \right) \)
$$j_0(x) = \frac{\sin x}{x}$$

$$j_1 = -x \int \frac{\sin x}{x^2}$$

$$n_0(x) = -\frac{\cos x}{x}$$

Note \( \frac{\cos x}{x} \to \infty \) as \( x \to 0 \)

but \( \frac{\sin x}{x} \to 1 \)

Therefore \( j_0(kr) \) is singular as \( r \to 0 \)

Choose \( B = 0 \)

$$U = Ar f j_0(kr)$$

Boundary condition at \( r = a \)

$$U(a) = 0 = j_0(ka)$$

$$k = \frac{1}{a} \beta_{nl}$$

$$\beta_{nl}$$ is \( n \)th zero of \( l \)th spherical Bessel function

$$E_{nl} = \frac{h^2}{2m^2} \beta_{nl}^2$$

indep. of \( m \)

$$N_{nlm}(r, \theta, \phi) = A_{nl} j_{l}(\beta_{nl}r/a) y_{lm}^{m}(\theta, \phi)$$

And from normalization
Finite spherical well (Prob. 4.9)

\[ V = \begin{cases} 0 & r \geq a \\ -V_0 & r < a \end{cases} \]

Look for \( n=0 \) solutions (lowest energy)

\[ \frac{d^2u}{dr^2} = -k^2u \quad r < a \]

\[ k = \sqrt{\frac{2m(E + V_0)}{h}} \]

Looks just like section 2.6 Finite Square Well outside \( r > a \)

\[ \frac{d^2u}{dr^2} = k^2u \quad r > a \]

bound state \( E < 0 \) but \( -V_0 < E < 0 \)

Only difference in 3D is b.c. at \( r = 0 \)

\[ \psi = 0 \quad \psi_r \rightarrow \psi_r \] at \[ r = 0 \]

So \( \psi \rightarrow 0 \) as \( r \rightarrow 0 \)

\[ \psi = A \sin kr \quad r < a \]

\[ B e^{-kr} \quad r > a \]

match \( \psi \) and \( \psi' \) at \( r = a \)

\[ \frac{\psi'}{\psi} = \frac{\psi'}{A \sin kr} \bigg|_{r=a} = -e^{-kr} \]
\[ k \cot ka = -j \]

In the limit of very weakly bound state \( E \to 0 \) from below,

\[ \cot ka \approx 0 \quad \text{or} \quad ka = \frac{\pi}{2} \]

\[ \frac{\hbar^2}{2m} = E + V_0 = V_0 \]

\[ V_0 = \frac{k^2}{2ma^2} \left( \frac{\pi^2}{4} \right) = \frac{\hbar^2 \pi^2}{8ma^2} \]

If \( V_0 \) is less than this \( \Rightarrow \) no bound state in 3 dim.

Example: Proton + neutron has one bound state (deuteron \( E \approx 0 \))

While \( p + n \) for two protons or two neutrons is slightly weak and disintegrates, or de neutron does not exist.

Note in 1 dim always a bound state \( \Rightarrow \) any attractive pot.

If \( p + p \) would bind to form \( 2 \) He, or \( 16 \) billion years.