Lecture 22  H Atom

Finite spherical well \( V=\frac{3}{2}V_0 \) \( r<a \), \( r>a \)

Look \( l=0 \) solutions

Inside \( \frac{d^2u}{dr^2} = \left[ \frac{l(l+1)}{r^2} - k'^2 \right] u \)

\( k' = \sqrt{\frac{2m}{\hbar}} \frac{(E+V_0)}{r} \)

For \( l=0 \) \( u = A \sin (k'r) \)

B.C. \( u(0) = 0 \)

Outside \( V=0 \) \( \frac{d^2u}{dr^2} = \left[ \frac{l(l+1)}{r^2} - k'^2 \right] u \)

\( k'^2 = \frac{2mE}{\hbar^2} < 0 \)

\( \alpha^2 = -k'^2 > 0 \)

\( E < 0 \) \( l=0 \)

\( N = B e^{-\alpha r} + D e^{\alpha r} \) not normalizable

Math \( \Psi' \) and \( \Psi \) at \( r=0 \)

\( \Psi' = A \sin k'a = B e^{-\alpha a} \)

\( k' \cos k'a = -\alpha B e^{-\alpha a} \)

\( \frac{\Psi'}{\Psi} = -\alpha \)

Trans. equation for \( E \rightarrow d(E) \) and \( k'(E) \)

What is weakest possible \( V_0 \) that has at least one bound state?

\( E \rightarrow 0 \) as \( V_0 \) decreases

So \( \frac{p^2}{2m} = E + V_0 = V_0 \) since \( E \rightarrow 0 \)

\( k'a = \frac{\pi}{2} \)
\[ k' = \frac{\pi}{2a} \]

\[ V_0 = \frac{\hbar^2 \pi^2}{8ma^2} \]

If \( V_0 \) is less than \( \frac{\hbar^2 \pi^2}{8ma^2} \), then the potential has no bound states.

Example interaction between two protons is attractive and short ranged. Can model it as spherically well. However, no \( \alpha \) system exists.

If \( ^2 \text{He} \) did exist, the Sun would burn via \( p + p \rightarrow ^2 \text{He} + \gamma \) in seconds, instead of 10 billion years.

\( p + p \rightarrow d + e^+ + \nu \)

because it involves the weak interaction.

**H Atom**

Nucleus at rest, Coulomb pot. with electron

\[ V = -\frac{e^2}{4\pi \varepsilon_0 r} \]

\[ H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} + V(r) \]

Radial eq.

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial r^2} + \left[-\frac{e^2}{4\pi \varepsilon_0 r} + \frac{k^2}{r^2} \right] \psi = E \psi \]
\[ J^2 = \frac{\sqrt{-2mE}}{\hbar} \]

\[ \frac{1}{J^2} \frac{d^2 U}{dp^2} = \left[ 1 - \frac{m e^2}{2 \pi \epsilon_0 k^2} \right] \frac{1}{x^r} + \frac{l(l+1)}{(x^r)^2} U \]

Define \[ P = Jx^r \] dimensionless

\[ P_0 = \frac{m e^2}{2 \pi \epsilon_0 k^2} \]

\[ \frac{d^2 U}{dp^2} = \left[ 1 - \frac{p_0}{p} + \frac{l(l+1)}{p^2} \right] U \]

Dimensionless radial equation

\[ N(r, \theta, \phi) = \frac{U(x^r)}{x^r} Y^m_l(\theta, \phi) \]

Full wave function \[ E = -\frac{\hbar^2}{2m} \frac{1}{x^r} \]

Look for bound state solution

General procedure

1. Find large \( p \) behavior

   \[ \frac{d^2 U}{dp^2} \approx U \] as \( p \to \infty \)

   \[ U = A e^{-p} + B e^{p} \]

2. Find small \( p \) behavior as \( p \to 0 \)

   \[ \frac{d^2 U}{dp^2} \approx \frac{l(l+1)}{p^2} U \]

   \[ U = p^{l+1} e^p \]
Check

\[ u = p^k \]
\[ u' = k(p^{k-1}) p^l \]
\[ u'' = l(l+k) p^{l-1} = ll(p^{l-1}) \]

Note that \( p^l \) blows up as \( p \to 0 \).

\( \square \)

Guess

\[ u(p) = p^{k+1} e^{-p} \]

This defines \( V(p) \). Expect \( V(p) \) to only depend weakly on \( p \) as \( p \to \infty \).

\( \square \)

Put \( \square \) in radial equation

\[ u' = \left[ \frac{l+1}{p} - 1 + \frac{1}{p} \right] p^{k+1} e^{-p} V \]
\[ u'' = p e^{-p} \left[ \frac{l(l+1)}{p} + p - 2l - 2 \right] V + 2(l+1-p) V' \]
\[ \frac{\partial V}{\partial p^2} + 2(\ln - p) \frac{\partial V}{\partial p} + \left[ p_0 - 2l - 1 \right] V = 0 \]

From \( \frac{\partial^2 V}{\partial p^2} = \left[ 1 - \frac{p_0}{p} + l(l+1) \right] V \)

\( \square \)

Power series for \( V(p) = \sum a_j p^j \)

\[ V' = \sum_{j=0}^{\infty} a_j p^{j-1} = \sum_{j=0}^{\infty} a_j j+1 \]

\( \square \)
\[ v'' = \sum_{j=0}^{\infty} j (j+1) a_{j+1} \rho^{-j} \]

\[ \sum_{j=0}^{\infty} j (j+1) a_{j+1} \rho^j + 2(j+1) \sum_{j=1}^{\infty} (j+1) a_{j+1} \rho^j \]

\[ - 2 \sum_{j=0}^{\infty} j a_j \rho^j + [\rho_c - 2(j+1)] \sum_{j=0}^{\infty} a_j \rho^j = 0 \]

Equate coeff of \( \rho^j \)

\[ E_j (j+1) + 2(j+1)(j+1) a_{j+1} + [\rho_c - 2(j+1) - 2j] a_j = 0 \]

\[ a_{j+1} = \frac{2(j+1) - \rho_c}{(j+1)(j+2l+2)} a_j \]

For large \( j \)

\[ a_{j+1} \approx \frac{2}{j+1} a_j \]

So

\[ a_j \approx \frac{2^j}{j!} A \]

\[ v(\rho) = A \sum_{j=0}^{\infty} \frac{2^j \rho^j}{j!} = A e^{2\rho} \]

\[ u(\rho) = A \rho^{l+1} e^{-\rho} e^{2\rho} = A \rho^{l+1} e^\rho \]

This blows up as \( \rho \to \infty \)

\[ \text{Need series to terminate} \]

For some \( j_{\text{max}} \)

\[ 2(j_{\text{max}}+l+1) - \rho_c = 0 \]
Then \( a_{j=0} \geq c \) so \( a_{j > j_{\text{max}} = 0} \)

\[ P_0 = 2\pi \quad \rho_0 = \frac{m e^2}{2\pi \varepsilon_0 h^2 \alpha} \]

\[ E = -\frac{\hbar^2 \alpha^2}{2m} \]

\[ E = -\frac{\hbar^2}{8m} \frac{m^2 e^4}{(2\pi)^2 \varepsilon_0^2 h^4 + 4\hbar^2} \frac{1}{2m} \]

\[ E_n = -\frac{mc^2}{2} \frac{\alpha^2}{n^2} \]

\[ mc^2 = 0.511 \text{ MeV} \quad \text{milli-electron volts} \]

\[ c = \text{speed of light} \]