Lecture 21 Radial Equation

$Y^m_l(\theta, \phi)$ is the same for all spherically symmetric potentials.

The shape of $V(r)$ only changes $R(r)$ and the allowed energies $E$

$$\frac{1}{2} \frac{d^2 r}{dr^2} + \frac{2mr^2}{h^2} \left( V(r) - E \right) R = \ell (\ell + 1) \frac{R}{r^2}$$

1c1. $R = \frac{u}{r}$

$$R' = \frac{1}{r} \left( u' - \frac{u}{r^2} \right)$$

$$\frac{d^2}{dr^2} R' = \frac{1}{r} \left( u'' - \frac{u''}{r^2} \right)$$

$$-\frac{h^2}{2m} \frac{d^2 u}{dr^2} + \left[ \frac{V}{2m} + \frac{\ell (\ell + 1)}{r^2} \right] u = Eu$$

Radial equation. This looks like an $\sin \theta$ eq. with a pot.

$V_{\text{eff}} = V(r) + \frac{h^2}{2m} \frac{\ell (\ell + 1)}{r^2}$

Contains an extra conf. Fugal term $\frac{h^2}{2mr^2}$ which describes how "angular mom. norm." keeps particles away from origin.

Normalizations:

$$\int_0^\infty r^2 \sin \theta \, d\theta \, d\phi \, |R(r)|^2 \left| \frac{\ell}{2} \frac{\ell + 1}{r} \right|^2 = 1$$

1c. $\int_0^\infty r^2 R(r)^2 = 1 - \int \int d\theta \, d\phi \, |Y^m_l|_r^2 \psi_i^2$
Infinite Spherical Well

To go further need form of $V(r)$

**Example**

\[ V(r) = \begin{cases} \frac{C}{r} & r < a \\ \infty & r > a \end{cases} \]

Outside \( V(r) \equiv 0 \quad r > a \)

Inside \( V = 0 \)

\[ \frac{d^2U}{dr^2} = \left[ \frac{l(l+1)}{r^2} - k^2 \right] U \]

\[ k = \sqrt{\frac{2mE}{\hbar^2}} \]

If \( l = 0 \) then

\[ \frac{d^2U}{dr^2} = -k^2 U, \quad U = A \sin(kr) + B \cos(kr) \]

BC. \( R(r) = U/r \) behaves as \( r \to 0 \)

thus \( U(0) = 0 \), not \( R(0) = 0 \)

also \( U(a) = 0 \)

\[ U = A \sin \left( \frac{nr}{a} \right) \]

Also \( \int_0^a U^2 = 1 \)

\( A^2 \propto \frac{1}{a} \)

\[ A = \sqrt{\frac{2}{\alpha}} \]

\[ R = \sqrt{\frac{2}{va}} \sin \left( \frac{nr}{a} \right) \frac{1}{r} \]

\[ Y_{l,m}^{n}(\theta,\phi) = \sqrt{\frac{2}{va}} \frac{1}{r} \sin \left( \frac{nr}{a} \right) Y_l^m(\theta,\phi) \]

\[ l = 0 \implies m = 0 \quad Y_0^0 = \frac{1}{\sqrt{4\pi}} \]
\[ p_0^0 = 1 \quad \Rightarrow \quad U_0^0 = A, \quad \text{constant} \]

\[ A^2 \sum_{\nu} \sin \nu \theta \cos \phi = 1 \]

\[ A^2 (1 - 1) 2 \pi = 1 \quad \Rightarrow \quad A = \frac{1}{\sqrt{4 \pi}} \]

\[ U_0^0 = \frac{1}{\sqrt{4 \pi}} \]

\[ n = 1, 2, 3, \ldots \]

\[ E_{n\ell} = \frac{\hbar^2 \ell^2}{2m} n^2 \]

\[ E_{n\ell} = \frac{\hbar^2 (n^2 - \ell^2)}{2m} \]

Allowed energies for \( l = 0 \) solutions which are spherically symmetric

Solution of \( \frac{\partial^2 u}{\partial r^2} + \left( \frac{l(l+1)}{r^2} - \frac{k^2}{a^2} \right) u = 0 \)

For \( l \neq 0 \)

\[ U(r) = A r^{\ell} j_\ell(kr) + B r^{\ell} n_\ell(kr) \]

\( j_\ell(kr) \) = spherical Bessel function of order \( \ell \)

\( n_\ell(kr) \) = spherical Neumann function of order \( \ell \)

\[ j_\ell(x) = (-\frac{x}{\ell})^\ell \left( \frac{1}{x} \frac{d}{dx} \right)^\ell \sin x \]
\[ j_0(x) = \frac{\sin x}{x} \quad \kappa_n = -\cos x/x \]

\[ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \]

\[ j_2(x) = \left( \frac{2}{x^2} - \frac{1}{x} \right) \sin x - \frac{3}{x^2} \cos x \]

\[ j_n(x) \rightarrow \frac{x^n}{(2n+1)!!}, \quad n \geq 1 \]

\[ F_0 = 1 \quad x \ll 1 \]

\[ R(r) = A j_n(kr) \]

Note for \( l = 0 \)

\[ R = \frac{\sin kr}{k} \]

agrees with what we have before

B.C. \( a^+ \quad r = a \quad R(a) = 0 \]

\[ j_n(\kappa_n a) = 0 \]

Let \( \beta_{nl} \) be the \( n \)th zero of \( \ell \)th spherical Bessel function

\[ \kappa_{nl} = \frac{1}{a} \beta_{nl} \]

\[ E_{nl} = \frac{k^2}{2m^2} \beta_{nl}^2 \]

\[ N_{nlm}(r, \theta, \phi) = A_{nl} j_\ell(\kappa_{nl} r) Y_{\ell m}(\theta, \phi) \]

\[ \frac{1}{4} \]
Example 2: Finite spherical well

\[ V = \begin{cases} \xi - V_0 & r < a \\ 0 & r \geq a \end{cases} \]

Look for \( l = 0 \) solutions. Inside:

\[ \frac{\partial^2 \Psi}{\partial r^2} = -k'^2 \Psi \]

\[ k'^2 = \sqrt{2m (E + V_0)} \]

B. Condition \( U(0) = 0 \)

Outside, if \( E < 0 \), \( N = e^{-\alpha r} \)

decaying exp

\[ N(r \to \infty) \to 0 \]

at \( r = a \) \( \Psi \) and \( \Psi' \) are

\[ \frac{\Psi'}{\Psi} = \frac{k' \cos k'a}{\sin k'a} = -\alpha \]

\[ k' = \sqrt{-2mE} \]

\[ \Psi'(\text{inside}) = \Psi(\text{outside}) \]

\[ \Psi(\text{inside}) = -\Psi(\text{outside}) \]