Lecture 6  Stationary States

Last time

To every observable \( \rightarrow \) operator

Example \( \hat{X} = x \rightarrow \hat{P} = -i\hbar \frac{\partial}{\partial x} \)

Expectation value \( \langle 0 \rangle = \int_{-\infty}^{\infty} \psi^* \psi \, dx \)

For Gaussian wave function \( \Delta x \Delta p = \hbar \)

In general \( \psi(x,t) \) depends on both time and position. To start solving \( \psi \), first look for special solutions

\[ \psi_i(x,t) = \psi_i(x) F(t) \]

which are simple products. Note capital \( \psi \) is function of \( x,t \), lower case \( \psi \) is function of \( x \)

Later we will find most general solution as superposition

\[ \psi(x,t) = \sum a_i \psi_i(x,t) \]

\( a_i = \) expansion coeff.

For now assume \( \star \) and drop label:

\[ -i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi \]

\[ -i\hbar \psi(x) \frac{\partial F}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi(x) F(t) \]
\[
- \frac{i}{\hbar} \frac{1}{f(t)} \frac{\partial^2 \Psi(x,t)}{\partial t^2} - \frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x) \Psi(x,t) = E \Psi(x,t)
\]

Only true if both sides equal a constant \( E \) independent of both \( x,t \)

\[
- \frac{i}{\hbar} \frac{dF}{dt} = E F(t)
\]

Time independent Schrödinger equation

\[
\frac{dF}{dt} = - \frac{iE}{\hbar} F(t)
\]

\[ F(t) = C e^{-\frac{iEt}{\hbar}} \]

Choose \( C = 1 \) can absorb it into \( \Psi \)

\[ F = e^{-\frac{iEt}{\hbar}} \]

\[ \Psi(x,t) = \Psi(x) e^{-\frac{iEt}{\hbar}} \]

Note \( F \Psi F^* = \Psi \) so above is called a stationary state. Wave function only has a trivial time dependence. Expectation values of time independent operators will be
Can write time independent S. eq.

\[ H = \frac{\hat{p}^2}{2m} + \hat{V} \]

with \( \hat{p} = -i\hbar \frac{d}{dx} \)

\[ \hat{V} = V(x) \]

and \( \hat{H} \psi(x) = E \psi(x) \)

It is called Hamiltonian \( \hat{H} = \hat{\mathbf{T}} + \hat{\mathbf{V}} \)

\[ \langle \hat{\mathbf{H}} \rangle = \int \psi^* \hat{\mathbf{H}} \psi \ dx = \int \psi^* \hat{\mathbf{T}} \psi \ dx + \int \psi^* \hat{\mathbf{V}} \psi \ dx = \int \psi^* \hat{\mathbf{H}} \psi \ dx = E \]

\[ \langle \hat{\mathbf{H}}^2 \rangle = \int \psi^* \hat{\mathbf{H}}^2 \psi \ dx = \int \psi^* \hat{\mathbf{T}}^2 \psi \ dx + \int \psi^* \hat{\mathbf{V}}^2 \psi \ dx = \int \psi^* \hat{\mathbf{H}}^2 \psi \ dx = E^2 \]

\[ \Delta E = \int \langle \hat{\mathbf{H}}^2 \rangle - \langle \hat{\mathbf{H}} \rangle^2 \ dx = E^2 - E^2 = 0 \]

No uncertainty in energy. State is eigenstate of energy.

To solve time independent S. eq.

Need to specify \( V(x) \)

**Example** Infinite Square Well

\[ V(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \infty & \text{otherwise} \end{cases} \]
If \( \Psi \neq 0 \) outside the well

\(< \hat{V} > \) and \(< \hat{H} > \) will be \( \infty \)

So finite energy solutions go to zero outside the well

\[ \Psi(x) = 0 \quad \text{when} \quad V = \infty \]

**Boundary conditions**

(a) Wave Function is always continuous

(b) \( \frac{\partial \Psi}{\partial x} \) is continuous except where

Thus \( \Psi(x) \) must go to zero as \( x \to 0 \) or \( x \to a \)

**S - eq inside well** \( V = 0 \)

\[ -\frac{1}{2m} \frac{d^2 \Psi}{dx^2} = E \Psi \]

\[ \frac{d^2 \Psi}{dx^2} = -k^2 \Psi \]

\( k = \) wave vector \[ = \sqrt{\frac{2mE}{h}} \]

Above is equation for simple harmonic motion

\[ \Psi = A \sin kx + B \cos kx \]

Most general solution \( A, B \) are undetermined constants.
Boundary conditions:

1. \( \psi(0) = 0 \Rightarrow B = 0 \)
   \[ \psi(x) = A \sin kx \]

2. \( \psi(a) = 0 = \sin ka \)

Note: \( A = 0 \) makes \( \psi \equiv 0 \) and cannot normalize. 
\[ \int_0^a \psi^2 \, dx = 1 \]

\[ k = \frac{n\pi}{a} \quad n = 1, 2, 3 \]

\( n = 0 \) gives \( \psi \equiv 0 \) again.

\[ E_n = \frac{k^2}{2m} = \frac{n^2\pi^2 k^2}{2ma^2} \]

This is a fundamental way QM works. Imposing boundary conditions forces energy to be quantized.

Particles in a box can only have discrete allowed energies.

Normalize:
\[ \int_0^a \psi^2 \, dx = \int_0^a |A|^2 \sin^2(kx) \, dx = \frac{|A|^2 a}{2} = 1 \]

\[ |A|^2 = \frac{2}{a} \]

\[ \psi_n = \sqrt{\frac{2}{a}} \sin \left( \frac{n\pi x}{a} \right) \]