Lecture 10 Delta-Function Pot. 1/31/01

Fourier Transforms and inverse F. transforms

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk \\ F(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, dx\, e^{-ikx} \end{aligned}}$$

$F(k)$ is F. transform of $f(x)$
$f(x)$ is inverse F. transform of $F(k)$

Only difference is sign of exp.

That the F. transform can be inverted is called Plancherel's theorem. It requires integrals to exist.

Free particle wave packet

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \phi(k)\, e^{ikx}$$

So

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, \Psi(x,0)\, e^{-ikx}$$

Now state with energy $E = \frac{\hbar^2 k^2}{2m}$ has a time dep. $e^{-iEt/\hbar}$

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \phi(k)\, e^{i\left(kx - \frac{\hbar k^2 t}{2m}\right)}}$$

$\Psi(x,0)$ given by inital conditions
F. transform of $\Psi(x,0)$ gives $\phi(k)$
Finally, time dep. given by last integral

We will see later that $\phi(k)$ is wave function in momentum space.

$|\Psi(x,t)|^2 \, dx$ is prob. to find particle between x and $x + dx$

$|\phi(k)|^2 \, dk$ is prob. to find particle with momentum between $\hbar k$ and $\hbar(k + dk)$

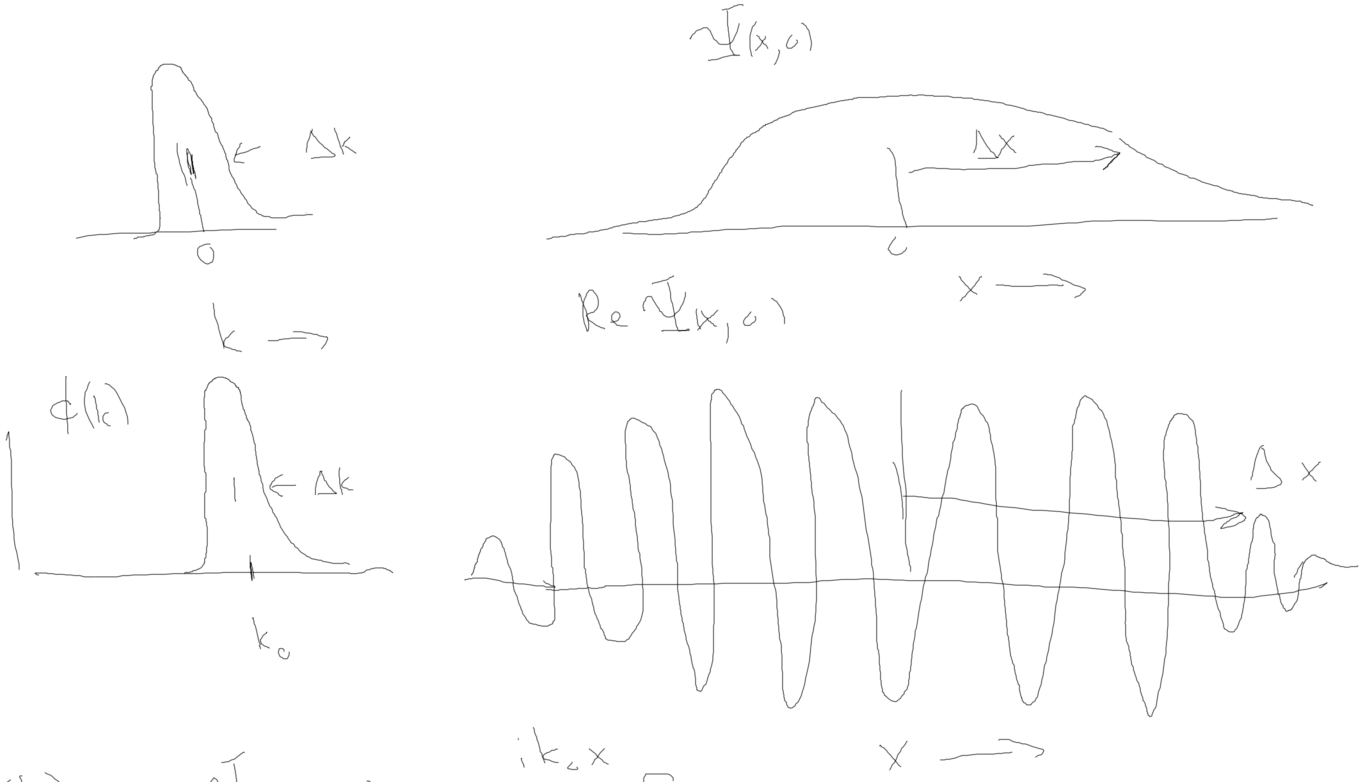
Normalization $\quad \int |\Psi(x,t)|^2 \, dx = 1$

$$\int |\phi(k)|^2 \, dk = 1$$

Time dep. wave packet

(a)

(b)



In (b) $\quad \Psi(x,0) \sim e^{ik_0 x} \times f(x)$

where $f(x)$ is a slowly varying function of x which describes the envelope of the wave packet and describes the uncert. Δx

The average momentum is

$$\langle p \rangle = \int dx\, \Psi^*(x,0)\, \frac{\hbar}{i}\frac{\partial}{\partial x} \Psi(x,0)$$

$$= \int dx\, f(x)^* e^{-ik_0 x} \frac{\hbar}{i}\frac{\partial}{\partial x} \left[e^{ik_0 x} f(x) \right]$$

$$= \frac{\hbar}{i} \int dx\, f^* e^{-ik_0 x} \left(ik_0 e^{ik_0 x} f + f' e^{ik_0 x} \right)$$

$$\simeq \hbar k_0 \int_{-\infty}^{\infty} dx\, f^* f = \boxed{\hbar k_0}$$

If f is a slowly varying function then $f' \ll ik_0 f$

So (b) desribes a wave packet moving with momentum $\sim \hbar k_0$

show p.mpg

Note in general two kinds of solutions

1. Bound states $E < V(|x| = \infty)$

   Energy spectrum is discrete and particle stays localized in space

   Example Infinite square well and H. Osc. have only bound states.

2. Scattering states $E > V(|x| = \infty)$

   Energy spectrum is cont. and

particle can escape to ∞

Example free particle.

Note many pot. have both bound and scattering states. Example delta-function which we will discuss now.

Delta - Function pot.

Dirac delta function

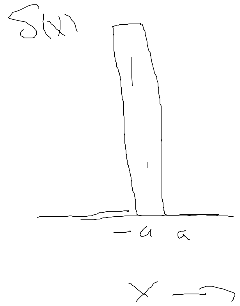
$$\delta(x) = \begin{matrix} 0 & \text{If } x \neq 0 \\ \infty & \text{If } x = 0 \end{matrix}$$

and $\int_a^b \delta(x)\,dx = 1$ if $a < 0 < b$

Example

$$\delta(x) = \begin{cases} \frac{1}{2a} & |x| < a \\ 0 & |x| > a \end{cases}$$

in limit as $a \to 0$



narrow, tall rectangle of unit area.

Since $\delta(x) = 0$ for $x \neq 0$, $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$

Conside $V = -\alpha\,\delta(x)$

greek alpha greek delta

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - \alpha\,\delta(x)\,\psi(x) = E\psi$$

This pot. yields both bound states $E<0$ and scattering states $E>0$.

1 Look for bound state solutions $E<0$

If $x \neq 0$ $\quad \frac{\partial^2\psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi = \kappa^2\psi$

Greek kappa

$\kappa = \frac{\sqrt{-2mE}}{\hbar}$ note $E<0$

$$\psi = Ae^{-\kappa x} + Be^{+\kappa x}$$

For $x<0$ $\quad \psi(x \to -\infty) \to 0$

$\psi = Be^{\kappa x} \quad x<0$

For $x>0$ $\quad \psi(x \to +\infty) \to 0$

$\psi = Ae^{-\kappa x} \quad x>0$

What are boundary conditions at $x=0$?

a) ψ is allways cont.

$\psi = Ae^{-\kappa|x|} \quad x<0$ or $x>0$

b) $\frac{\partial\psi}{\partial x}$ is cont. except if $V=\infty$

But $V=\infty$ at $x=0$ so $\frac{\partial\psi}{\partial x}$ is not cont.

To find discont. in $\frac{\partial \psi}{\partial x}$ try integrating Schrodinger eq. from $-\epsilon$ to $+\epsilon$ and take limit $\epsilon \to 0$.

$$\int_{-\epsilon}^{\epsilon} dx \left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}\right) + \int_{-\epsilon}^{\epsilon} dx \left[-\alpha\,\delta(x)\,\psi(x)\right]$$

$$= \int_{-\epsilon}^{\epsilon} dx \; E\psi$$

$$-\frac{\hbar^2}{2m}\left[\frac{\partial \psi}{\partial x}(+\epsilon) - \frac{\partial \psi}{\partial x}(-\epsilon)\right] - \alpha\,\psi(0) \approx 2\epsilon E\psi(0)$$

$= 0$ as $\epsilon \to 0$

$$\boxed{\left.\frac{\partial \psi}{\partial x}\right|_{+\epsilon} - \left.\frac{\partial \psi}{\partial x}\right|_{-\epsilon} = -\frac{2m\alpha}{\hbar^2}\,\psi(0)}$$

If $\psi = Ae^{-\kappa|x|}$

$$\left.\frac{\partial \psi}{\partial x}\right|_{+\epsilon} = -\kappa A \qquad \left.\frac{\partial \psi}{\partial x}\right|_{-\epsilon} = +\kappa A$$

$$-2\kappa A = -\frac{2m\alpha}{\hbar^2} A$$

$$\kappa = \frac{m\alpha}{\hbar^2} \qquad E = -\frac{\hbar^2\kappa^2}{2m} = -\frac{\hbar^2 m^2\alpha^2}{\hbar^4\, 2m}$$

$$\boxed{E = -\frac{m\alpha^2}{2\hbar^2}}$$