1/29/01

Lecture 9 Free Particle

Last time: Harmonic Osc.

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{k}{2}x^2\right]\psi = E\psi$$

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}\, x \qquad \omega = \sqrt{\frac{k}{m}}$$

dimensionless length

$$K = 2E/\hbar\omega$$

dimensionless energy

$$\frac{\partial^2\psi}{\partial\xi^2} = (\xi^2 - K)\psi$$

$$\psi = h(\xi)\, e^{-\xi^2/2}, \Rightarrow \qquad h'' - 2\xi h' + (K-1)h = 0$$

$$h = \sum_{i=0}^{\infty} a_i \xi^i, \Rightarrow \qquad a_{i+2} = \left[\frac{2i+1-K}{(i+2)(i+1)}\right] a_i$$

Series must terminate for wave function to be normalizable

$$\boxed{E_n = (n+\tfrac{1}{2})\hbar\omega} \qquad \Rightarrow \quad K = 2n+1$$

Energy eigenvalues $n = 0, 1, 2, \ldots$

$n=0$ $\quad h = a_0 \quad \psi_0 = a_0 e^{-\xi^2/2}$

$n=1$ $\quad h = a_1\xi \quad \psi_1 = a_1 \xi e^{-\xi^2/2}$

$n=2$ $\quad h = a_0(1-2\xi^2) \quad \psi_2 = a_0(1-2\xi^2)e^{-\xi^2/2}$

Normalize to find a_0 or a_1

Define Hermite polynomials $H_n(\xi)$

(1) n^{th} order polynomial in ξ

(2) $$a_{i+2} = \frac{2i+1-(2n+1)}{(i+1)(i+2)} a_i$$

(3) Coef. of ξ^n, $a_n = 2^n$

$H_0 = 1$

$H_1 = 2\xi$

$H_2 = 4\xi^2 - 2$

$H_3 = 8\xi^3 - 12\xi$

$H_4 = 16\xi^4 - 48\xi^2 + 12$

Note problem 2.18

$$H_{n+1} = 2\xi H_n - 2n H_{n-1}$$

... See course lectures \ computer programs

Normalize Harmonic osc. wave function $\int_{-\infty}^{\infty} dx\, \psi_n^*(x)\, \psi_n(x) = 1$

$$\boxed{\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi)\, e^{-\xi^2/2}}$$

$E_n = (n+\frac{1}{2})\hbar\omega$, $\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{k}{2}x^2\right]\psi_n = E_n \psi_n(x)$

Free Particle

Seems like $V=0$ would be most simple example. However it is very subtle.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi \qquad \frac{\partial^2\psi}{\partial x^2} = -k^2\psi$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$\psi(x) = Ae^{ikx} + Be^{-ikx}$

Problem with b.c. $\psi(x \to \pm\infty) \to 0$
Otherwise wave function is not normalizable

Solution is to make a <u>wave packet</u>

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dx$$

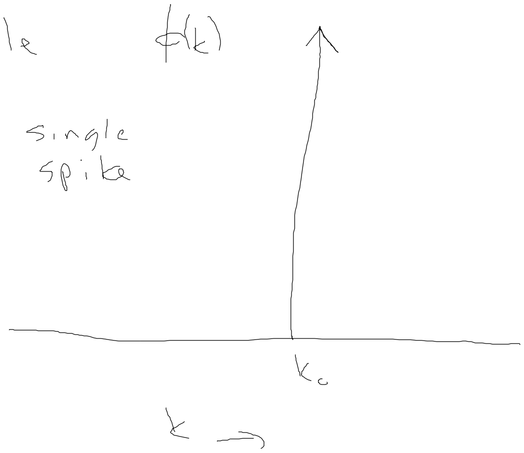
This wave function is <u>normalizable</u>, see this later (in problem 2.22 for example).

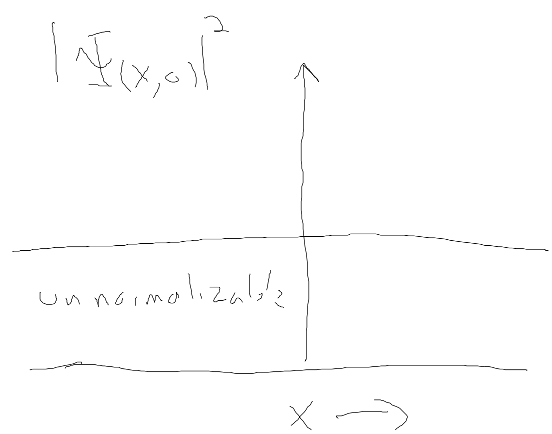
Wave function does not have a fixed energy. Instead it is made up of a range of wave vectors k and energies

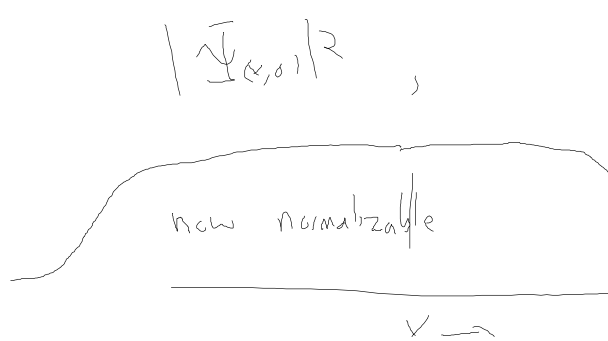
$$E_k = \frac{\hbar^2 k^2}{2m}$$

$\Rightarrow \Delta E > 0$

Example







Fourier Transforms and inverse F. transforms

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk \\ F(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \end{aligned}}$$

$F(k)$ is F. transform of $f(x)$
$f(x)$ is inverse F. transform of $F(k)$
[only difference is sign of exp.]

Note this is called Plancherel's theorem (That F. transform can be inverted) and requires that integrals exist.

So
$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx}\, dx$$

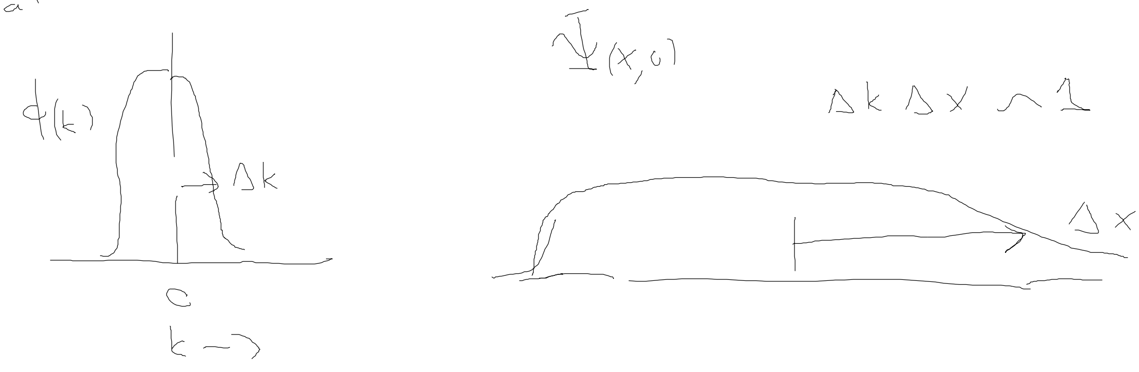
Finally $\Psi_k(x,t) = \Psi(x,0)\, e^{-iE_k t/\hbar}$
with $E_k = \frac{\hbar^2 k^2}{2m}$

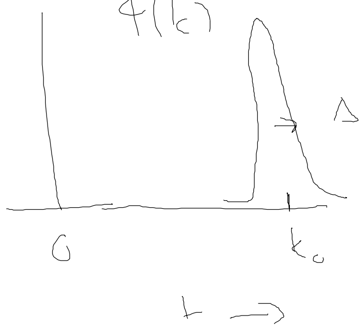
$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k)\, e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}\, dk}$$

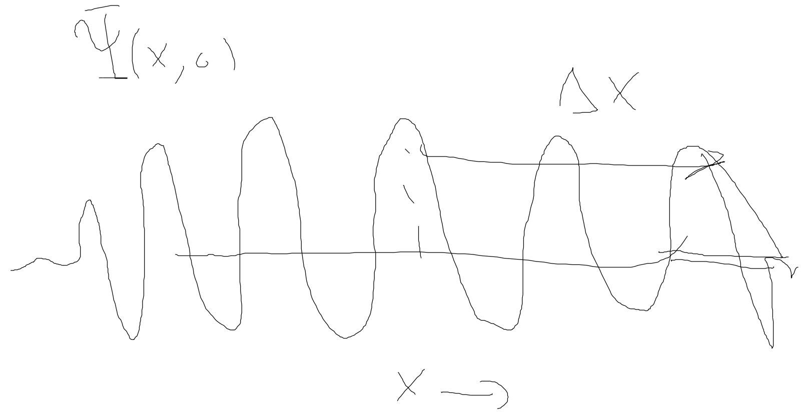
Time dep. wave packet

In general

$\Delta k\, \Delta x \sim 1$







Plot ρ.mpg