1/26/01

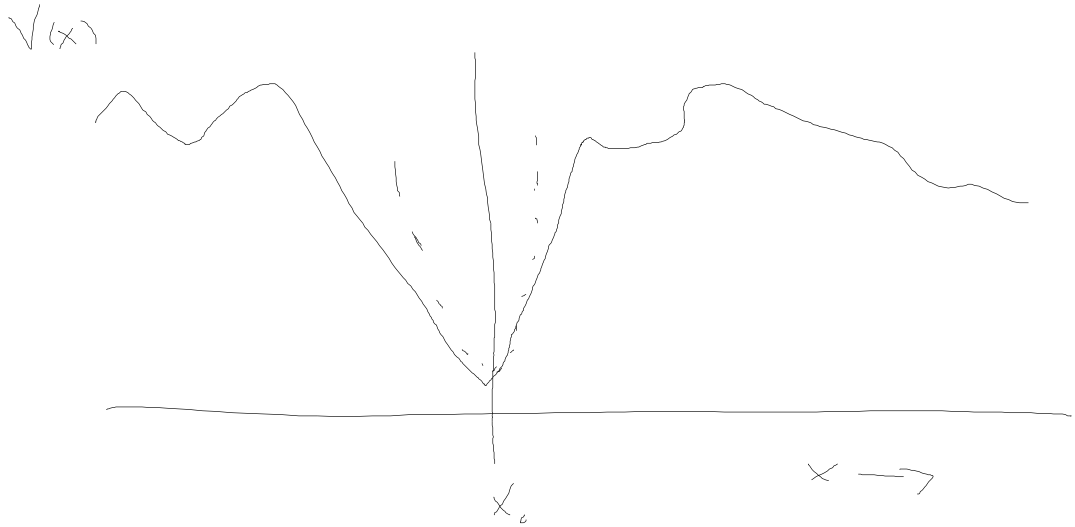
Lecture 8 Harmonic Osc.

Consider $F = -kx$ (Hooke's Law)

or

$$V = \frac{1}{2} k x^2$$

Note almost any pot. is $\cong$ harmonic near a minimum



$$V \cong V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2} V''(x_0)(x - x_0)^2 + \ldots$$

minimum $V' = 0$

$$k = V''(x_0)$$

can approx. any pot. $\cong$ as harmonic osc. near minimum

Classical frequency $\omega = \sqrt{\frac{k}{m}}$

$$x = A \sin \omega t + B \cos \omega t$$

Schrodinger eq. (time indep.)

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) + \frac{k}{2} x^2 \psi(x) = E \psi(x)$$

Simplify notation

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x \qquad \text{Greek xi}$$

ξ is Dimensionless length $\quad \left[\sqrt{\frac{m\omega}{\hbar}}\right] = \frac{1}{L}$

IE $\quad [\hbar] = m\frac{L}{t} L \qquad [\omega] = \frac{1}{T}$

K is dimensionless energy $\equiv 2E/\hbar\omega$

$$[\hbar\omega] = m\frac{L^2}{T}\frac{1}{T} = [E]$$

$$\boxed{\frac{\partial^2\psi}{\partial\xi^2} = (\xi^2 - K)\psi}$$

Schrod. Eq. in dimensionless form.

General procedure

(1) Extract large ξ behavior of ψ

(2) Expand rest of ψ in power series

(3) Demand series terminates so wave function is normalizable. $\Rightarrow$ Gives quantized E

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(1) Look at eq. for large ξ, $\xi^2 \gg K$

$$\frac{\partial^2\psi}{\partial\xi^2} \approx \xi^2\psi$$

approx. solution $\quad \psi = Ae^{-\xi^2/2} + Be^{\xi^2/2}$

Second term blows up so $B = 0$.

$\psi \propto e^{-\xi^2/2}$

$\psi' = -\xi e^{-\xi^2/2}$

$\psi'' = (\xi^2 - 1) e^{-\xi^2/2} \approx e^{-\xi^2/2} \xi^2 = \xi^2 \psi$ ✓

(2) Guess $\psi(\xi) = h(\xi) e^{-\xi^2/2}$

Assume $h(\xi)$ is a slow or mild function of ξ at large ξ

$\psi' = h' e^{-\xi^2/2} + h(-\xi) e^{-\xi^2/2}$

$\psi'' = [h'' - 2\xi h' + (\xi^2 - 1) h] e^{-\xi^2/2}$

$$\boxed{\frac{\partial^2 h}{\partial \xi^2} - 2\xi \frac{dh}{d\xi} + (K - 1) h = 0}$$

Schrodinger eq. rewritten for h

Equiv. to $\frac{\partial^2 \psi}{\partial \xi^2} = (\xi^2 - K)\psi$

Expand $h = a_0 + a_1 \xi + a_2 \xi^2 + \ldots$

$$\boxed{h = \sum_{j=0}^{\infty} a_j \xi^j}$$ Power series for h

$$h' = \sum_{j=1}^{\infty} a_j \, j \, \xi^{j-1} = a_1 + 2a_2 \xi + 3a_3 \xi^2 + \ldots$$

$$h'' = \sum_{j=2}^{\infty} a_j \, j(j-1) \xi^{j-2} = 2a_2 + 6a_3 \xi + 12 a_4 \xi^2 + \ldots$$

Rewrite h'': let $k = j - 2 \Rightarrow j = k + 2$

$$h'' = \sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) \xi^k$$

Put this into S. eq. for h

$$h'' - 2\xi h' + (K-1)\,h = 0$$

$$\sum_{k=0}^{\infty} a_{k+2}\,(k+2)(k+1)\,\xi^k$$

$$-2\sum_{j=1}^{\infty} a_j\, j\, \xi^j \qquad + \sum_{j=0}^{\infty} a_j \xi^j (K-1) = 0$$

No harm to include $j=0$

If eq. is to be true for all ξ
Coef. of each power of ξ must vanish

$$\sum_{j=0}^{\infty}\left[a_{j+2}(j+2)(j+1) - 2a_j j + (K-1)a_j\right]\xi^j = 0$$

or

$$\boxed{a_{j+2} = \left[\frac{2j+1-K}{(j+2)(j+1)}\right] a_j}$$

Recursion relation for a_{j+2} given a_j

If series does not terminate $\Rightarrow$ for large j

$$a_{j+2} \simeq \frac{2}{j}\,a_j$$

or $$a_j = \frac{C}{(j/2)!}$$

$$h(\xi) = C\sum_j \frac{1}{(j/2)!}\,\xi^j = C\sum_k \frac{1}{k!}(\xi^2)^k$$

let $k = j/2$ $\quad j = 2k$

$$h = C e^{\xi^2} \qquad \psi = C e^{\xi^2} e^{-\xi^2/2}$$

$\psi = C e^{\xi^2/2}$ This is not normalizable!

Note have to work hard to keep the bad $e^{\xi^2/2}$ large ξ behavior from coming back.

Chose K so that $a_{n+2} = 0$ for some n

$$\boxed{K = 2n+1} = 2E/\hbar\omega$$

With this choice series terminates and $a_{n+4} = a_{n+6} = \dots = 0$

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega}$$

Energy eigenvalues of simple harmonic osc.

Example $n=0$

$a_2 = 0 \cdot a_0 = 0$ $\qquad E_0 = \frac{1}{2}\hbar\omega$

also chose $a_1 = 0$ so $a_3 = a_5 \dots = 0$

$h_0 = a_0$ $\qquad \psi(\xi) = a_0 e^{-\xi^2/2}$

$n=1$ chose $a_0 = 0 \longrightarrow a_2 = a_4 = 0$

$a_3 = 0 \cdot a_1 = 0 = a_5 = \dots$

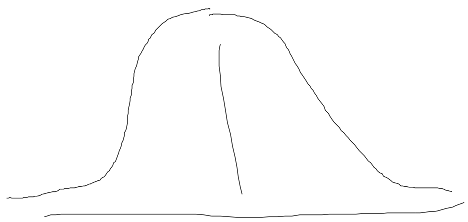
$\psi(\xi) = a_1 \xi e^{-\xi^2/2}$ $\qquad E_1 = \frac{3}{2}\hbar\omega$

$n=2$ $\qquad a_4 = 0 \cdot a_2 = 0$, $\quad K = 5$

$$a_2 = \left(\frac{1-K}{2}\right) a_0 = -2a_0$$

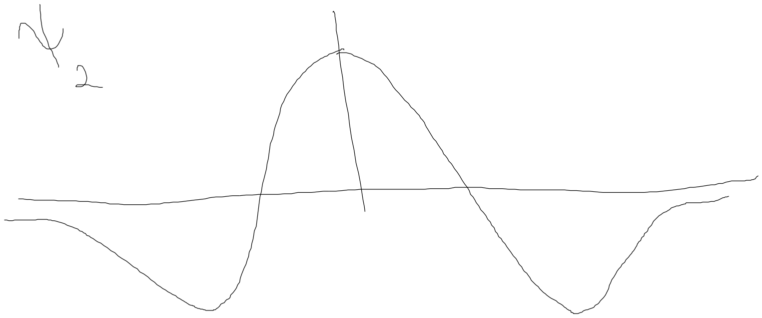
$h = a_0(1 - 2\xi^2)$ $\qquad \psi = a_0(1-2\xi^2)e^{-\xi^2/2}$

ψ_0 -



ψ_1

ψ_2



Note Hermite polynomials $H_n(\xi)$ satisfy coef. of ξ^n is 2^n

$H_0 = 1$

$H_1 = 2\xi$

$H_2 = 4\xi^2 - 2$

$H_3 = 8\xi^3 - 12\xi$

$H_4 = 16\xi^4 - 48\xi^2 + 12$

...

Normalize $\int dx\, \psi_n^*(x)\psi_n(x) = 1$

$$\boxed{\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi)\, e^{-\xi^2/2}}$$

$E_n = (n+\frac{1}{2})\hbar\omega$

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{k}{2}x^2\right]\psi_n = E_n\psi_n$$