1/22/01

Lecture 6 Stationary States

Start reading Chapter 2 sections 2.1 + 2.2

Look for a simple class of solutions to Schrodinger eq.

$$\Psi(x,t) = \psi(x) f(t) \qquad (*)$$

(Seperation of variables)

Equation (*) is not the most general solution of the Schrodinger eq. However it is a very simple and usefull first step.

$$-i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi$$

Substitute (*) into S. eq. and divide by $\Psi(x,t)$

$$\frac{1}{f(t)}(-i\hbar)\frac{df(t)}{dt} = \frac{1}{\psi(x)}\left(-\frac{\hbar^2}{2m}\right)\frac{d^2\psi(x)}{dx^2} + V(x)$$

The left hand side is only a function of t while R.H.S. is only a function of x. This will only work for all x and all t if both sides are constants. Call the constant E

$$E = \frac{1}{f}(-i\hbar)\frac{df}{dt} \Rightarrow \boxed{-i\hbar \frac{df}{dt} = E f(t)}$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)}$$

Time indep. S. equation

$$\boxed{f = Ce^{-iEt/\hbar}} \qquad C = \text{constant}$$

$$i\hbar \frac{df}{dt} = E f \quad \checkmark$$

Chose normalization constant to normalize $\psi(x)$ so set $C=1$

$$\boxed{f = e^{-iEt/\hbar}}$$

$$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$$

Note $f^* f = 1$ so

$$|\Psi(x,t)|^2 = \psi^*(x)\,\psi(x)$$

Prob. density is independent of time. Therefore wave functions that can be written in the form of Eq. ⊛

$$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$$

are known as stationary states.

Define $\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}$ Hamiltonian operator

with $\hat{V} = V(x)$ and $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

Time indep. Schrodinger eq

$$\boxed{\hat{H}\psi(x) = E\,\psi(x)}$$

What is expectation value of energy

$$\langle E \rangle = \int dx\, \Psi^*(x,t)\, \hat{H}\, \Psi(x,t)$$

for a stationary state

$$\langle E \rangle = \int_{-\infty}^{\infty} dx\ \psi^*(x)\ \hat{H}\ \psi(x) = E \int_{-\infty}^{\infty} dx\ \psi^*(x)\psi(x)$$

but $\hat{H}\psi = E\psi$

$$\langle E \rangle = E$$

What is uncert. in energy

$$\Delta E = [\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2]^{1/2}$$

$$\langle \hat{H}^2 \rangle = \int dx\ \psi^* \hat{H}\hat{H}\psi = E \int dx\ \psi^* \hat{H}\psi$$

$$= E^2$$

so

$$\Delta E = [E^2 - E^2]^{1/2} = 0$$

Stationary states have zero uncertainty in energy. Every measurement of energy will yield same value ~~E~~ E with no spread.

How can things move in QM if stationary states have no time dep.?

Answer, general solution of time dep. S. eq. is not of form

$$\Psi_i(x,t) = \psi_i(x)\ e^{-iE_it/\hbar}$$

but it can be written as a <u>superposition</u>

$$\Psi(x,t) = \sum_{i=1}^{\infty} c_i\ \psi_i(x)\ e^{-iE_it/\hbar}$$

because this sum contains terms with more then one energy

$$\Psi^*(x,t)\,\Psi(x,t) = \sum_i c_i^* \psi_i^*(x)\, e^{+iE_it/\hbar} \sum_j c_j \psi_j(x)\, e^{-iE_jt/\hbar}$$

$$= \sum_i \sum_j c_i^* c_j\, \psi_i^*(x)\psi_j(x)\, e^{i(E_i-E_j)t/\hbar}$$

$$\neq \sum_i \sum_j c_i^* c_j\, \psi_i^*(x)\, \psi_j(x)$$

Here the c_i are constants indep. of position and time.

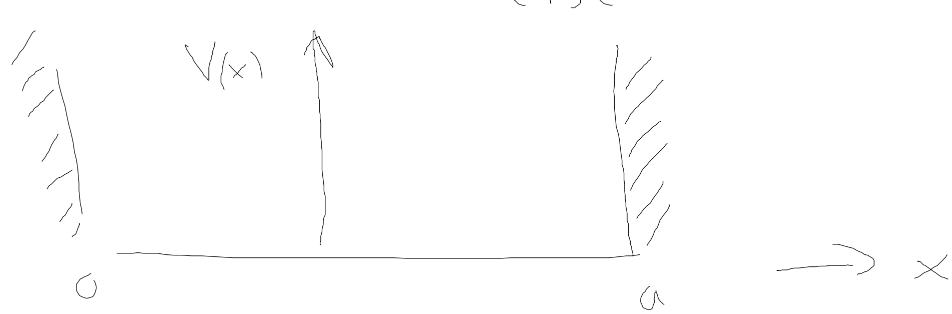
We now study the time indep. Schrodinger eq. (Often just Schrodinger eq.)

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x)} \qquad (1)$$

For some simple potentials $V(x)$

<u>Infinite Square Well</u>

$$V(x) = \begin{cases} 0, & 0 \le x \le a \\ \infty & \text{else} \end{cases}$$



<u>In general we solve eq. (1) subject to boundary conditions</u>

(1) If $\psi(x)$ is to be normalized

$\psi \to 0$ as $|x| \to \infty$

(2) $\psi(x)$ is always cont.

(3) $\frac{d\psi}{dx}$ is cont. except where $V \to \infty$

(4) Finite energy solutions requies

$\psi \to 0$ where $V \to \infty$

For square well $\psi(0) = \psi(a) = 0$

Inside $V = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$$

or

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

Solution is $\psi = A\sin kx + B\cos kx$

$\psi(0) = 0 \longrightarrow B = 0$

$\psi(a) = 0 = A\sin k\cancel{a}$

Can't set $A = 0$ non normalizable so

$\sin ka = 0 \Rightarrow ka = 0, \pm\pi, \pm 2\pi, \pm 3\pi$

$$\boxed{k_n = \frac{n\pi}{a}} \qquad n = 1, 2, 3, \ldots$$

$$\boxed{E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2 \pi^2 \hbar^2}{2 m a^2}}$$

$$\psi_n = A \sin\left[\left(\frac{n\pi}{a}\right) x\right]$$

Normalize

$$1 = \int_0^a dx \, A^2 \sin^2\left(\frac{n\pi}{a} x\right) = \frac{1}{2} a A^2$$

$$A = \sqrt{\frac{2}{a}} \qquad \boxed{\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)}$$