1/19/01

Lecture 5 Uncertainty Principle

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x)\; x\; \Psi(x)\; dx$$

$$\langle p \rangle = m\frac{d}{dt}\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x) \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi(x)\; dx$$

In general repesent a classical quantity Q with an operator $\hat{Q}$

Operator $\equiv$ takes one function and gives you another one

Example $\frac{d}{dx}$ takes $f(x) \longrightarrow \frac{df}{dx}$

$\hat{x} = x$ multiplication by x

$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

In general any function of x and p

$$Q(x,p) \longrightarrow \hat{Q} = Q\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)$$

So

$$\langle Q(x,p) \rangle = \int \Psi^* \underbrace{Q\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)}_{\hat{Q}} \Psi\; dx$$

Put operator $\hat{Q}$ in between Ψ^* and Ψ

Example: Kinetic energy

$$T = \frac{p^2}{2m} \longrightarrow \hat{T} = \frac{\hat{p}^2}{2m} = \frac{1}{2m}\frac{\hbar}{i}\frac{\partial}{\partial x}\frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \qquad 2^{nd} \text{ derivative}$$

$$\langle T \rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial x^2}\; dx$$

Example Prob. 1.9 (a)

Let P_{ab} be probability to find particle between a and b

$$P_{ab} = \int_a^b \Psi^*(x,t)\,\Psi(x,t)\,dx$$

Calculate how P_{ab} changes with time

$$\frac{d}{dt}P_{ab} = \int_a^b \left[\frac{\partial \Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial \Psi}{\partial t}\right] dx$$

From last time using S. eq.

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{iV}{\hbar}\Psi$$

$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{iV}{\hbar}\Psi^*$$

$$\frac{d}{dt}P_{ab} = -\frac{i\hbar}{2m}\int_a^b \left[\frac{\partial^2 \Psi^*}{\partial x^2}\Psi - \Psi^*\frac{\partial^2 \Psi}{\partial x^2}\right] dx$$

Integrate by parts

$$\int_a^b \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\,dx = \frac{\partial \Psi^*}{\partial x}\Psi\Big|_a^b - \int_a^b \frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x}\,dx$$

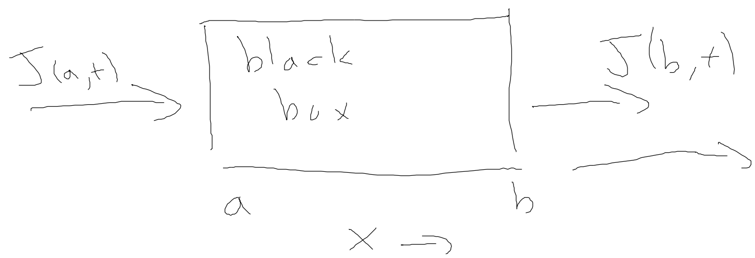
$$-\int_a^b \Psi^*\frac{\partial^2 \Psi}{\partial x^2}\,dx = -\Psi^*\frac{\partial \Psi}{\partial x}\Big|_a^b + \int_a^b \frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x}\,dx$$

$$\int_a^b \left[\frac{\partial^2 \Psi^*}{\partial x^2}\Psi - \Psi^*\frac{\partial^2 \Psi}{\partial x^2}\right] dx = \left[\frac{\partial \Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial \Psi}{\partial x}\right]\Big|_a^b$$

Let $$J(x,t) \equiv \frac{i\hbar}{2m}\left[\Psi(x)\frac{\partial \Psi^*(x)}{\partial x} - \Psi^*(x)\frac{\partial \Psi(x)}{\partial x}\right]$$

$$\frac{d}{dt} P_{ab} = -\frac{i\hbar}{2m}\left[\frac{\partial \Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial \Psi}{\partial x}\right]\Bigg|_a^b$$

$$= J(a,t) - J(b,t)$$



Change in # of particles inside box is prob. current going in $J(a,t)$ minus prob. current going out $J(b,t)$

So $$J(x,t) = \frac{i\hbar}{2m}\left[\Psi\frac{\partial \Psi^*}{\partial x} - \Psi^*\frac{\partial \Psi}{\partial x}\right]$$

Describes the # of particles per unit time that flow past the point x.

Dimensions $\int \Psi^*\Psi \, dx = 1$

$$[\Psi] = L^{-1/2}$$

$$[\hbar] = \text{mass}\,\frac{L}{\text{time}}\,L = [\Delta x][\Delta p]$$

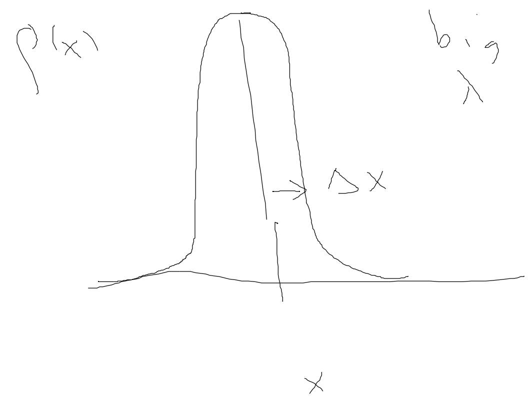
$$[J] = \frac{L^2\,\text{mass}}{\text{mass}\;\text{time}}\;\frac{1}{L}\;\frac{1}{L^{1/2}}\;\frac{1}{L^{1/2}} = \frac{1}{\text{time}}$$

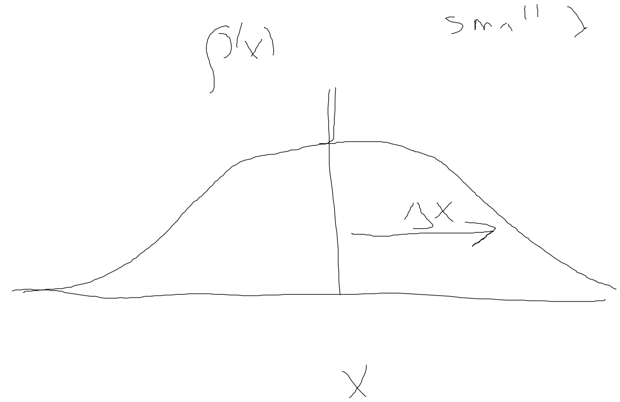
Example related to 1.6

Let $\Psi = A^{1/2} e^{-\frac{\lambda}{2}x^2}$

$$\rho = \Psi^* \Psi = A\, e^{-\lambda x^2}$$

calculate $\langle p^2 \rangle$, $\langle p \rangle$ and Δp





You will show $\Delta x = \sigma \propto \lambda^{-1/2}$

Note Units $[\lambda] = \frac{1}{L^2}$ so $[\lambda^{-1/2}] = L$

A big implies a small uncertainty in x

$$\langle p \rangle = \int_{-\infty}^{\infty} A^{1/2} e^{-\frac{\lambda}{2}x^2} \frac{\hbar}{i}\frac{\partial}{\partial x} A^{1/2} e^{-\frac{\lambda}{2}x^2}\, dx$$

$$= \frac{A\hbar}{i}(-\lambda) \int_{-\infty}^{\infty} e^{-\lambda x^2}\, x\, dx = 0$$

odd integral

$$\langle p^2 \rangle = -A\hbar^2 \int_{-\infty}^{\infty} e^{-\frac{\lambda}{2}x^2} \frac{\partial^2}{\partial x^2}\left(e^{-\frac{\lambda}{2}x^2}\right) dx$$

$$\frac{\partial^2}{\partial x^2} e^{-\frac{\lambda}{2}x^2} = \frac{\partial}{\partial x}(-\lambda x) e^{-\frac{\lambda}{2}x^2} = (\lambda^2 x^2 - \lambda) e^{-\frac{\lambda}{2}x^2}$$

$$\langle p^2 \rangle = -A\hbar^2 \lambda \int_{-\infty}^{\infty} (\lambda x^2 - 1) e^{-\lambda x^2}\, dx$$

$$t = \lambda^{1/2} x$$

$$dx = dt / \lambda^{1/2}$$

$$\langle p^2 \rangle = -\hbar^2 \lambda A \int_{-\infty}^{\infty} (t^2 - 1) e^{-t^2} \frac{dt}{\lambda^{1/2}}$$

Normalization

$$1 = A \int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \frac{A}{\lambda^{1/2}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

let $t = \lambda^{1/2} x$

$$A = \lambda^{1/2} \Big/ \int_{-\infty}^{\infty} dt\, e^{-t^2}$$

$$\langle p^2 \rangle = \hbar^2 \lambda \underbrace{\left[\frac{\int_{-\infty}^{\infty} (1 - t^2) e^{-t^2} dt}{\int_{-\infty}^{\infty} e^{-t^2} dt} \right]}_{\text{indep. of } \lambda}$$

$$\Delta p = \left[\langle p^2 \rangle - \langle p \rangle^2 \right]^{1/2} = \langle p^2 \rangle^{1/2}$$

$$\propto \hbar \lambda^{1/2}$$

Now product $\Delta p \, \Delta x \propto \hbar \lambda^{1/2} \left(\frac{1}{\lambda^{1/2}} \right)$

$$= \hbar$$

Product is indep. of λ. Can chose λ to make Δp or Δx small but not both!