1/17/01

# Lecture 4 Time Dependance

## Review

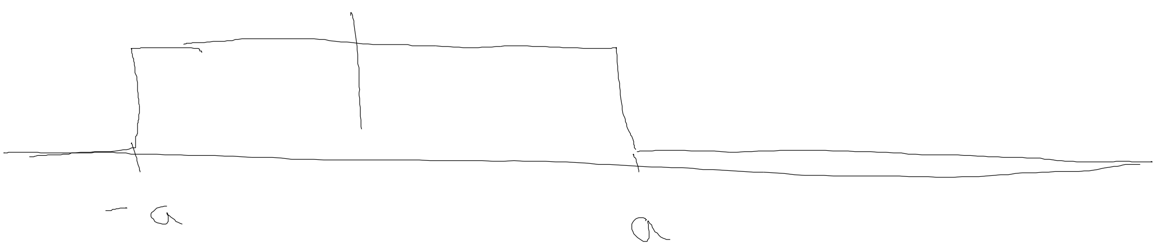
Normalization: $\int_{-\infty}^{\infty} dx \; \Psi^*(x,t)\, \Psi(x,t) = 1$

Expectation value
Mean Value
Average value : $\langle x \rangle = \int_{-\infty}^{\infty} dx \; x \, \Psi^* \Psi$

Variance : $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$

Standard deviation $\Delta x = \sigma = \left[ \langle x^2 \rangle - \langle x \rangle^2 \right]^{1/2}$

Example: $\Psi(x) = \begin{cases} C & |x| < a \\ 0 & x > a \end{cases}$




Normalize: $\int_{-\infty}^{\infty} dx \; \Psi^* \Psi = \int_{-a}^{a} c^* c \; dx = |c|^2 \int_{-a}^{a} dx$

$$1 = |c|^2 \, 2a \Rightarrow c = \frac{1}{\sqrt{2a}}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} dx \; x \, \Psi^* \Psi = \int_{-a}^{a} dx \; x \, |c|^2$$

$$= |c|^2 \left. \frac{x^2}{2} \right|_{-a}^{a} = 0$$

odd Function

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx \; x^2 \, \Psi^* \Psi = |c|^2 \int_{-a}^{a} dx \, x^2$$

$$= |c|^2 \left. \frac{x^3}{3} \right|_{-a}^{a} = \frac{2a^3}{3} \frac{1}{2a} = \frac{a^2}{3}$$

$$\Delta x = \sigma = [\langle x^2\rangle - \langle x\rangle^2]^{1/2} = \frac{a}{\sqrt{3}} = .577\,a$$

Time dependance

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi$$

If $\Psi(x,t)$ is normalized at $t=0$ will it stay normalized at late times?

Calculate

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx \;\Psi^*(x,t)\,\Psi(x,t)$$

$$= \int_{-\infty}^{\infty} dx \left[\frac{\partial \Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial \Psi}{\partial t}\right]$$

$$\frac{\partial \Psi}{\partial t} = \frac{1}{i\hbar}\;-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + \frac{V}{i\hbar}\Psi$$

$$\frac{1}{i} = -i$$

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{iV}{\hbar}\Psi$$

Take complex conjugate $i \to -i$

$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{iV}{\hbar}\Psi^*$$

Note assume $V(x) = V^*(x) \Rightarrow$ Pot. Real.

$$\frac{d}{dt}\int dx\,\Psi^*\Psi = \int dx\left[-\frac{i\hbar}{2m}\left(\frac{\partial^2 \Psi^*}{\partial x^2}\right)\Psi + \frac{i\hbar}{2m}\Psi^*\frac{\partial^2 \Psi}{\partial x^2}\right]$$

$$+\frac{i}{\hbar}\int dx\left[V\Psi^*\Psi - V\Psi^*\Psi\right] \stackrel{?}{=} \frac{d}{dt}1$$

$$\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \Psi \frac{\partial^2 \Psi^*}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x}\right)$$

$$\frac{d}{dt}\int dx\, \Psi^* \Psi = \frac{i\hbar}{2m}\int dx\, \frac{\partial}{\partial x}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x}\right) \quad \text{(A)}$$

$$= \frac{i\hbar}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x}\right)\Bigg|_{-\infty}^{\infty}$$

$$= 0$$

If $\Psi$ is normalized then

$\Psi \to 0$ as $x \to \pm\infty$

So

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\, |\Psi(x,t)|^2 = 0$$

If wave function is normalized at $t=0$ it stays normalized at late time

## Velocity and Momentum

Start with $\langle x \rangle = \int_{-\infty}^{\infty} x\, \Psi^* \Psi\, dx$

Calculate "velocity"

$$\frac{d}{dt}\langle x \rangle = \int_{-\infty}^{\infty} x \frac{\partial}{\partial t}(\Psi^* \Psi)\, dx$$

$$= \int dx\; x \left(\frac{\partial \Psi^*}{\partial t}\Psi + \Psi^* \frac{\partial \Psi}{\partial t}\right)$$

We have simply added a factor of x to equation (A)

$$\frac{d}{dt}\langle x\rangle = \frac{i\hbar}{2m}\int_{-\infty}^{\infty} dx \; x \; \frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)$$

Integrate by parts

$$= -\frac{i\hbar}{2m}\int dx \left(\frac{\partial}{\partial x}x\right)\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)$$

$$+ \frac{i\hbar}{2m}\, x\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)\Bigg|_{-\infty}^{\infty}$$

$\cancel{\partial x}$ $\frac{\partial x}{\partial x} = 1$ and Boundary term $\to 0$ because wave function $\to 0$ to be normalized

$$\frac{d}{dt}\langle x\rangle = -\frac{i\hbar}{2m}\int_{-\infty}^{\infty} dx \left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)$$

~~integrate~~ integrate 2nd term by parts and it becomes = to first term

$$\boxed{\frac{d}{dt}\langle x\rangle = -\frac{i\hbar}{m}\int_{-\infty}^{\infty} dx \; \Psi^*\frac{\partial\Psi}{\partial x}}$$

$$\langle p\rangle = m\langle v\rangle = m\frac{d}{dt}\langle x\rangle$$

$$\boxed{\langle p\rangle = \int_{-\infty}^{\infty} dx \; \Psi^* \frac{\hbar}{i}\frac{\partial\Psi}{\partial x}}$$

represent observable O by operators $\hat{O}$

$$\hat{x} = x \qquad\qquad \boxed{\langle O\rangle \equiv \int_{-\infty}^{\infty} dx \; \Psi^* \hat{O}\Psi}$$

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

Consider any classical quantity that is a function of $x$ and $p$

$$\langle Q(x,p)\rangle = \int \Psi^* \, Q\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right) \Psi \, dx$$

Example kinetic energy

$$T = \frac{p^2}{2m} = \frac{1}{2} m v^2$$

$$\langle T \rangle = \left\langle \frac{\hat{p}^2}{2m} \right. = \frac{1}{2m}\int_{-\infty}^{\infty} \Psi^* \frac{\hbar}{i}\frac{\partial}{\partial x}\frac{\hbar}{i}\frac{\partial}{\partial x} \Psi \, dx$$

$$= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial x^2} \, dx$$

Note two $p \longrightarrow$ involve $2^{nd}$ derivative

$$\langle V(x) \rangle = \int \Psi^* \, V(x) \, \Psi \, dx$$

Pot. energy is a function of $x$ not $p$ so no derivatives

Total energy $E = T + V$

$$\langle E \rangle = \langle T \rangle + \langle V \rangle$$

$$= \int_{-\infty}^{\infty} \Psi^* \left[ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \right] \Psi \, dx$$

Uncert. principle involves $\Delta x$ and $\Delta p$

$$\Delta p = \left[ \langle p^2 \rangle - \langle p \rangle^2 \right]^{1/2}$$

$$\langle p^2 \rangle = \int \Psi^* \frac{\hbar}{i}\frac{\partial}{\partial x}\frac{\hbar}{i}\frac{\partial}{\partial x} \Psi \, dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial x^2} \, dx$$