2/23/01

Lec. 19 Uncertainty Principle

$$\sigma_A^2 = \langle f|f\rangle \qquad |f\rangle \equiv (\hat{A} - \langle A\rangle)|\Psi\rangle$$

$$\sigma_B^2 = \langle g|g\rangle \qquad |g\rangle \equiv (\hat{B} - \langle B\rangle)|\Psi\rangle$$

$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2 \geq |\mathrm{Im}\langle f|g\rangle|^2$$

$$= \left[\frac{1}{2i}\left(\langle f|g\rangle - \langle g|f\rangle\right)\right]^2 = \left[\frac{1}{2i}\left(\langle \hat{A}\hat{B}\rangle - \langle \hat{B}\hat{A}\rangle\right)\right]^2$$

so
$$\boxed{\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\langle [\hat{A},\hat{B}]\rangle\right)^2}$$

Example $A = x \quad B = p$

$$[\hat{x},\hat{p}]f(x) = x\frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}(x f(x)) = -\frac{\hbar}{i}f$$

$$\Rightarrow \boxed{[\hat{x},\hat{p}] = i\hbar}$$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i}\langle i\hbar\rangle\right)^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$ original H.U.P. as special case.

Any two operators which do not commute correspond to <u>incompatible</u> observab[les]

<u>Energy Time Uncert.</u>

Problem with $\Delta t\, \Delta E \geq \hbar/2$
Time is not an observable in QM. Can't make a measurement of the time. Instead it is a parameter.

Instead think of Δt as the minimom time for an observable to change by a significant fraction of itself.

Calculate time rate of change of some quantity Q

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\langle \Psi | Q \Psi\rangle$$

$$= \langle \frac{\partial \Psi}{\partial t} | Q \Psi\rangle + \langle \Psi | \frac{\partial Q}{\partial t} \Psi\rangle + \langle \Psi | Q \frac{\partial \Psi}{\partial t}\rangle$$

S. Eq. $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$

$$\frac{d}{dt}\langle Q\rangle = -\frac{1}{i\hbar}\langle H\Psi | Q\Psi\rangle + \langle \Psi | \frac{\partial Q}{\partial t}\Psi\rangle + \frac{1}{i\hbar}\langle \Psi | Q H \Psi\rangle$$

$$\boxed{\frac{d}{dt}\langle Q\rangle = \langle \frac{\partial Q}{\partial t}\rangle + \frac{i}{\hbar}\langle [H, Q]\rangle}$$

Consider uncert. in Q and uncert. in E

$$\sigma_H^2 \sigma_Q^2 \geq \left(\frac{1}{2i}\langle[\hat{H}, \hat{Q}]\rangle\right)^2$$

assume $\frac{\partial Q}{\partial t} = 0$ $\quad \langle [H, Q]\rangle = \frac{\hbar}{i}\frac{d}{dt}\langle Q\rangle$

$$\sigma_H^2 \sigma_Q^2 \geq \frac{\hbar^2}{4}\left(\frac{d}{dt}\langle Q\rangle\right)^2$$

Define

$$\Delta t \equiv \sigma_Q \Big/ \frac{d}{dt}\langle Q\rangle$$

minimum time for Q to change by a detectable amount.

$$\boxed{\Delta t \, \sigma_H \geq \frac{\hbar}{2}}$$

$$\sigma_H^2 \left[\frac{\sigma_Q^2}{\left(\frac{d}{dt}\langle Q\rangle\right)^2}\right] \geq \frac{\hbar^2}{4}$$

QM in momentum space (Prob. 3.51)

Calculate expectation value of $\langle x\rangle$ in momentum space

$$\langle x\rangle = \int dx \, \Psi^*(x,t) \, x \, \Psi(x,t)$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp \, \Phi(p,t) \, e^{ipx/\hbar}$$

$$\Psi^*(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp' \, \Phi^*(p',t) \, e^{-ip'x/\hbar}$$

$$\langle x\rangle = \int dp' \, \Phi^*(p',t) \int dp \left[\frac{1}{2\pi\hbar}\int dx \, x \, e^{i(p-p')x/\hbar}\right.$$

Note problem 2.25

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} \, dk$$

If $k = 0$ integral gives ∞ if $k \neq 0$ integral osc. to death.

$$F(p',p) \equiv \int \frac{dx}{2\pi\hbar}\; x \; e^{i\left(\frac{p-p'}{\hbar}x\right)}$$

$$= \int \frac{dx}{2\pi\hbar}\left(-\frac{\hbar}{i}\frac{\partial}{\partial p'}\right) e^{i\left(\frac{p-p'}{\hbar}x\right)}$$

$$= \left(-\frac{\hbar}{i}\frac{\partial}{\partial p'}\right)\int \frac{dt}{2\pi}\; e^{i(p-p')t}$$

$$= \left(-\frac{\hbar}{i}\frac{\partial}{\partial p'}\right)\delta(p-p')$$

$$\langle x\rangle = \int dp \int dp'\; \Phi^*(p',t)\left(-\frac{\hbar}{i}\frac{\partial}{\partial p'}\right)\delta(p-p')\,\Phi(p,t)$$

$$\boxed{\langle x\rangle = \int dp'\; \Phi^*(p',t)\left(-\frac{\hbar}{i}\frac{\partial}{\partial p'}\right)\Phi(p',t)}$$

In momentum space

$$\hat{x} \longrightarrow \left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)$$

$$\hat{p} \longrightarrow p$$

$$\langle p\rangle = \int dp\; p\,|\Phi(p,t)|^2$$

Commutator

$$[\hat{x},\hat{p}]\,f(p) = -\frac{\hbar}{i}\frac{\partial}{\partial p}\,p f(p) + p\frac{\hbar}{i}\frac{\partial}{\partial p}f(p) = -\frac{\hbar}{i}f(p)$$

$$\Rightarrow \boxed{[\hat{x},\hat{p}] = i\hbar}$$ Same as in coordinate space.

so $Q \longrightarrow \hat{Q}\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}, t\right)$

or $\longrightarrow \hat{Q}\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}, p, t\right)$

In either chase

$$\sigma_x \, \sigma_p \geq \frac{\hbar}{2}$$

because of $[\hat{x}, \hat{p}] = i\hbar$