2/21/01

Lec. 18 Generalized Stat. Interpretation

Example of eigenstate

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

Eigenstates have zero uncert.

$$\sigma_E^2 = \langle(\hat{H} - \langle\hat{H}\rangle)^2\rangle = \langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2$$
$$= E^2 - E^2$$

3' If you measure an observable Q on a particle in the state $|\Psi\rangle$, you are certian to get one of the eigenvalues of $\hat{Q}$. (Example energy of H osc. must be $(N+\frac{1}{2})\hbar\omega$) The probability of getting the particular eigenvalue λ is equal to the absolute square of the λ component of $|\Psi\rangle$ when expressed in the orthonormal basis of eigenvectors.

Two kinds of eigenvectors

a) Discrete spectrum (H. Osc., particle in a box...

$$\hat{Q}\,|e_n\rangle = \lambda_n\,|e_n\rangle \qquad n = 1, 2, 3, \ldots$$

$$\langle e_n|e_m\rangle = \delta_{nm}$$

$$|\Psi\rangle = \sum_{n=1}^{\infty} c_n |e_n\rangle$$

$$|c_n|^2 = |\langle e_n|\Psi\rangle|^2 \qquad \text{prob. to get } \lambda_n$$

b) Continuous spectrum

$$\hat{Q}\,|e_k\rangle = \lambda_k\,|e_k\rangle \qquad -\infty < k < \infty$$

Delta function norm

$$\langle e_k | e_l \rangle = \delta(k-l)$$

$$|\Psi\rangle = \int_{-\infty}^{\infty} c_k |e_k\rangle \, dk$$

$$c_k = \langle e_k | \Psi \rangle$$

Prob. to get k between k and k + dk

$$\boxed{|c_k|^2 \, dk = |\langle e_k | \Psi \rangle|^2 \, dk}$$

Example position eigenstates

$$e_{x'}(x) = \delta(x - x')$$

$$c_{x'} = \langle e_{x'} | \Psi \rangle = \int \delta(x - x') \Psi(x,t) \, dx$$

$$= \Psi(x', t)$$

Prob. to find particle between x' and $x' + dx'$

$$|c_{x'}|^2 dx' = |\Psi(x', t)|^2 \, dx'$$

Example: momentum eigenstates

$$e_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \qquad -\infty < p < \infty$$

The p component of $|\Psi\rangle$ is

$$c_p = \langle e_p | \Psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t) \, dx$$

$$= \Phi(p, t)$$ momentum space wave function

$$|c_p|^2 dp = |\Phi|^2 dp$$ prob. to have mom. between p and p + dp

The state of the system is described by

$$|\Psi\rangle$$

This is an abstract vector in the Hilbert space. You chose a representation position space, momentum space, ... and deal with wave functions wich are dot products of $|\Psi\rangle$ with appropriate basis vectors

$$\Psi(x,t) \equiv \langle e_x|\Psi\rangle$$

$$\Phi(p,t) \equiv \langle e_p|\Psi\rangle$$

...

Uncertainty Principle

Consider any observable A

$$\sigma_A^2 = \langle(\hat{A} - \langle A\rangle)\Psi|(\hat{A} - \langle A\rangle)\Psi\rangle$$

$$= \langle\hat{A}^2\rangle - \langle A\rangle^2$$

from $\langle\Psi|(\hat{A} - \langle A\rangle)^2\Psi\rangle$ and $\hat{A} - \langle A\rangle$ is Hermitian

$$|f\rangle \equiv (\hat{A} - \langle A\rangle)|\Psi\rangle$$

$$\sigma_A^2 = \langle f|f\rangle$$

$$\sigma_B^2 = \langle g|g\rangle \quad \text{with} \quad |g\rangle \equiv (\hat{B} - \langle B\rangle)|\Psi\rangle$$

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2$$

$$|z|^2 = \mathrm{Re}\, z^2 + \mathrm{Im}\, z^2 \geq \mathrm{Im}\, z^2 = \left[\frac{1}{2i}(z - z^*)\right]^2$$

let $\quad z = \langle f | g \rangle$

$$\sigma_A^2 \sigma_B^2 \geq \left\{ \frac{1}{2i} \left[\langle f | g \rangle - \langle g | f \rangle \right] \right\}^2$$

$$\langle f | g \rangle = \langle (A - \langle A \rangle) \Psi | (B - \langle B \rangle) \Psi \rangle$$

$$= \langle \Psi | (A - \langle A \rangle)(B - \langle B \rangle) \Psi \rangle$$

$$= \langle \hat{A} \hat{B} \rangle - \langle A \rangle \langle B \rangle + \langle A \rangle \langle B \rangle - \langle A \rangle \langle B \rangle$$

$$= \langle \hat{A} \hat{B} \rangle - \langle A \rangle \langle B \rangle$$

$$\langle g | f \rangle = \langle \hat{B} \hat{A} \rangle - \langle A \rangle \langle B \rangle$$

so $\quad \langle f | g \rangle - \langle g | f \rangle = \langle \hat{A} \hat{B} \rangle - \langle \hat{B} \hat{A} \rangle$

Commutator $\quad [A, B] = [AB - BA]$

$$\boxed{\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2}$$

Generalized Uncertainty Principle

Example $\quad A = x \qquad B = p$

$$[\hat{x}, \hat{p}] = \left[x \frac{\hbar}{i} \frac{\partial}{\partial x} - \frac{\hbar}{i} \frac{\partial}{\partial x} x \right]$$

$$[\hat{x}, \hat{p}] f(x) = x \frac{\hbar}{i} f' - \frac{\hbar}{i} \frac{\partial}{\partial x} (x f)$$

test function

$$= -\frac{\hbar}{i} f$$

$$[\hat{x}, \hat{p}] = -\frac{\hbar}{i} = i\hbar$$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} \langle \hbar i \rangle \right)^2 = \left(\frac{\hbar}{2} \right)^2$$

$$\Rightarrow \quad \boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$