2/19/01

Lecture 17 Hermitian Transformations

Last time change of basis

$$|f_j\rangle = \sum_{i=1}^{N} S_{ij} |e_i\rangle$$

$$|\alpha\rangle = \sum_j a_j^e |e_j\rangle = \sum_j a_j^f |f_j\rangle$$

$$a_i^f = \sum_j S_{ij} a_j^e$$

Inner products

$$b^+ a = \sum_j b_j^{e*} a_j^e \quad \text{in } e \text{ basis}$$

$$= \sum_j b_j^{f*} a_j^f \quad \text{in } f \text{ basis}$$

$$b^{f+} = b^{e+} S^+ \quad \text{if} \quad \underline{b}^f = \underline{\underline{S}}\, \underline{b}^e$$

Remember $b^+ = \tilde{b}^*$

$$\underline{b}^{f+} \underline{a}^f = \underline{b}^{e+} \underline{\underline{S}}^+ \underline{\underline{S}}\, \underline{a}^e \quad , \quad \underline{a}^f = \underline{\underline{S}}\, \underline{a}^e$$

If transformation is unitary $S^+ = S^{-1}$ then

$$S^+ S = 1 \quad \text{and} \quad \underline{b}^{f+} \underline{a}^f = \underline{b}^{e+} \underline{a}^e$$

Inner product is independent of basis

Matrix elements are also independent of basis

$$\underline{b}^+ \underline{\underline{H}}\, \underline{a} = \underline{b}^{f+} \underline{\underline{H}}^f \underline{a}^f$$

$$= \cancel{b^{e+}} S^+ S \underline{\underline{H}}^e S^{-1} S a^e = \underline{b}^{e+} \underline{\underline{H}}^e \underline{a}^e$$

Where $\underline{\underline{H}}^f = S \underline{\underline{H}}^e S^{-1}$

Eigenvectors

$$\hat{T}\,|\alpha\rangle = \lambda\,|\alpha\rangle$$

Hermitian transformation

$$T^{\dagger} \equiv \tilde{T}^{*}$$

$$\langle T^{\dagger}\alpha|\beta\rangle = \langle\alpha|T\beta\rangle$$

If $T^{\dagger} = T$ than T is Hermitian or
if

$$\langle T\alpha|\beta\rangle = \langle\alpha|T\beta\rangle$$

Hermitian transformations have important properties

(1) Eigenvalues of Hermitian transformations are real.

$$\hat{T}|\alpha\rangle = \lambda\,|\alpha\rangle$$

$$\langle\alpha|T\alpha\rangle = \lambda\,\langle\alpha|\alpha\rangle = \lambda$$

But $\langle\alpha|T\alpha\rangle = \langle T\alpha|\alpha\rangle$ if T
Hermitian

$$\langle T\alpha| = \left[\overbrace{T|\alpha\rangle}^{=\ \lambda|\alpha\rangle}\right]^{*} = \lambda^{*}\,|\alpha\rangle^{\dagger} = \lambda^{*}\langle\alpha|$$

$$\langle\alpha|T\alpha\rangle = \langle T\alpha|\alpha\rangle = \lambda^{*}\langle\alpha|\alpha\rangle$$

Hence $\lambda^{*} = \lambda$ if T is Hermitian

(2) The eigenvectors of a Hermitian transformation for different eigenvalues are orthogonal

$$T|\alpha\rangle = \lambda\,|\alpha\rangle \qquad T|\beta\rangle = \mu\,|\beta\rangle$$

$$\mu \neq \lambda$$

$\langle \alpha | T\beta \rangle = \mu \langle \alpha | \beta \rangle$

$\langle \alpha | T\beta \rangle = \langle T\alpha | \beta \rangle = \lambda^* \langle \alpha|\beta \rangle = \lambda \langle \alpha | \beta \rangle$

$\mu \langle \alpha|\beta \rangle = \lambda \langle \alpha | \beta \rangle \Rightarrow (\mu - \lambda) \langle \alpha|\beta \rangle = 0$

but $\mu \neq \lambda$ so $\langle \alpha | \beta \rangle = 0$

(3) The eigenvectors of a Hermitian transform span space.

Consider a space of dimension N

$N \times N$ det $\Rightarrow \det[T - \lambda I] = 0$

So there are N eigenvalues.

If all N are distinct have N orthogonal eigenvectors which span space.

If λ occurs twice $\Rightarrow$ Have two eigenvectors which may not be orthogonal However you can make them orthogonal with Gram-Schmidt process. (See prob. 3.4)

Gives total of N orthogonal vectors which span space.

Function Spaces

Think of a vector space where the vectors are functions

(1) Sum of two functions is a function

(2) Product $cf(x)$ is another function.

Inner product

$$\langle f | g \rangle \equiv \int dx \; f^*(x) \; g(x)$$

Note Need integral to exist

$$\int dx \; |f|^2 < \infty \quad \text{and} \quad \int dx \; |g|^2 < \infty$$

<u>Example</u> set of all square integrable functions is an infinite dimensional vector space.

<u>Example</u> Set $P(N)$ of all polynomials of degree $< N$ on the interval $-1 \leq x \leq 1$

Note need to consider finite interval <u>because</u>

$$\int dx \; P(x)^2 = \infty$$

Operators as linear transformations

Operators such as x or d/dx take a function into another function

$$\hat{T} f(x) = \lambda \; f(x)$$

Call $f(x)$ an eigenfunction of $\hat{T}$

A Hermitian operator is one where

$$\langle f | \hat{T} g \rangle = \langle \hat{T} f | g \rangle$$ for all f, g in space.

Note $\hat{D} = \dfrac{d}{dx}$ is not Hermitian

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$$\langle f | \hat{D} g \rangle = \int_a^b f^* \frac{\partial g}{dx} \, dx$$

$$= f^* g \Big|_a^b - \int \frac{df^*}{d\cancel{x}} g \, dx$$

$$= f^* g \Big|_a^b - \langle \hat{D} f | g \rangle$$

To make it Hermitian

① Multiply by i to get rid of minus sign

② Only consider functions where

$$f(a) = f(b) \quad , \quad g(a) = g(b)$$

For example $a = -\infty$, $b = +\infty$ and $f, g \to 0$ as $x \to \pm\infty$ so that $\int_{-\infty}^{\infty} dx \, f^* f < \infty$

So for square integrable functions on $-\infty$ to ∞, $i\hat{D}$ is Hermitian

Hilbert Space

A complete inner product space is called a Hilbert space.

A complete space includes all of its limits. Example $P(\infty)$ is set of all polynomials and includes

$$f_N(x) = 1 + x + \frac{x^2}{2} + \dots \frac{x^N}{N!}$$

For finite N. It does not include e^x which is limit of f_N as $N \to \infty$

We are interested in set of square integrable functions from a, b

$L_2(a,b)$

and in particula $L_2(-\infty,\infty)$

Indeed to physicists Hilbert space is practically synonymous with $L_2(-\infty,\infty)$

Generalized Statistical Interpretation of QM

1. The state of a particle is represented by a normalized vector $|\Psi\rangle$ in the Hilbert space L_2.

2. Observable quantities $Q(x,p,t)$ are represented by Hermitian operators $\hat{Q}\left(x, \frac{\hbar}{i}\frac{d}{dx}, t\right)$; the expectation value of Q in $|\Psi\rangle$ is $\langle \Psi | \hat{Q} \Psi \rangle$

3. A measurement of Q on a particle in the state $|\Psi\rangle$ is certian to return the value λ iff $|\Psi\rangle$ is an eigenstate of $\hat{Q}$ with eigenvalue λ.