2/14/01

Lecture 16 Formalism Cont

Linear transformation

$$\hat{T}|e_j\rangle = \sum_{i=1}^{n} T_{ij}|e_i\rangle$$

$$\hat{T}|\alpha\rangle = |\alpha'\rangle \qquad |\alpha\rangle = \sum_i a_i|e_i\rangle$$

$$|\alpha'\rangle = \sum_{i=1}^{N} a_i'|e_i\rangle$$

$$\boxed{a_i' = \sum_{j=1}^{N} T_{ij}\, a_j}$$

Can work with matrices

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad n \times 1 \text{ column matrix}$$

$$\underline{\underline{T}} = n \times n \text{ square matrix} \quad \begin{pmatrix} T_{11} & T_{21} & \cdots & T_{n1} \\ T_{12} & & & T_{n2} \\ \vdots & & & \\ T_{1n} & & & T_{nn} \end{pmatrix}$$

Transpose of a column matrix is a row matrix

$$\tilde{a} = (a_1, a_2 \cdots, a_n)$$

Hermitian Conjugate or adjoint is transposed complex conjugate

$$\underline{\underline{T}}^{\dagger} \equiv \widetilde{T}^{*}$$

$$\underline{a}^{\dagger} = (a_1^*, a_2^*, \ldots a_n^*)$$

$$\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Inner product $\langle \alpha | \beta \rangle = \underline{a}^{\dagger} \underline{b}$

$$= (a_1^*, a_2^* \ldots a_n^*) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Matrix multiplication is not in general commutative

$$[\underline{\underline{S}}, \underline{\underline{T}}] \equiv ST - TS$$

$$(\underline{\underline{S}}\,\underline{\underline{T}})^{\dagger} = \underline{\underline{T}}^{\dagger} \underline{\underline{S}}^{\dagger}$$

Unit matrix $\underline{\underline{I}} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$ $I_{ij} = \delta_{ij}$

Inverse

$$\underline{\underline{T}}^{-1} \underline{\underline{T}} = \underline{\underline{T}}\,\underline{\underline{T}}^{-1} = I$$

A matrix has an inverse iff (if and only if) its determ. is nonzero

$$T^{-1} = \frac{1}{\det T} \widetilde{C}$$

C_{ij} = matrix of cofactors

$= (-1)^{i+j}$ det of matrix without i^{th} row and j^{th} column.

Example

$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det T = ad - bc$

$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ $\quad \tilde{C} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$T^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$T^{-1} T = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$\frac{1}{ad-bc} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

<u>Unitary</u> if inverse equals adjoint

if $U^{\dagger} = U^{-1}$ then matrix U is unitary

<u>Change of basis</u>

Important (prove QM is indep. of choice of basis)

Start with one orthonormal basis $|e_i\rangle$
Express vectors a and transformations T with $|e_i\rangle$

$$|\alpha\rangle = (a_1^e, a_2^e, \dots a_n^e)$$

Chose a new orthonormal basis $|f_i\rangle$

The basis vectors $|f_i\rangle$ can be written in terms of the old $|e_i\rangle$ because the $|e_i\rangle$ form a basis

$$|f_j\rangle = \sum_{i=1}^{N} S_{ij} |e_i\rangle \qquad j = 1, 2, \cdots n$$

$$\hat{S}|\alpha\rangle = \sum_j a_j^e \hat{S} |e_j\rangle$$

$$\hat{S}|e_j\rangle = \sum_i S_{ij} |e_i\rangle = |f_j\rangle$$

$$\hat{S}|\alpha\rangle = \sum_j a_j^e \sum_i S_{ij} |e_i\rangle = \sum_i a_i' |e_i\rangle$$

$$a_i' = \sum_j S_{ij} a_j^e = a_i^f$$

a_i^f are components of $|\alpha\rangle$ w.r.t. basis $|f_i\rangle$

$$\underline{a}^f = \underline{\underline{S}}\, \underline{a}^e$$

Let

$$\underline{a}^{e\prime} = \underline{\underline{T}}^e \underline{a}^e$$

be linear transformation in basis $|e_i\rangle$

$$\underline{a}^e = \underline{\underline{S}}^{-1} \underline{a}^f$$

$$\underline{a}^{f\prime} = \underline{\underline{S}}\, \underline{a}^{e\prime} = \underline{\underline{S}}\, \underline{\underline{T}}^e \underline{a}^e$$

$$= \underline{\underline{S}}\, \underline{\underline{T}}^e \underline{\underline{S}}^{-1} \underline{a}^f$$

So $\boxed{\underline{\underline{T}}^f = \underline{\underline{S}}\, \underline{\underline{T}}^e \underline{\underline{S}}^{-1}}$ linear trans in new basis related to linear trans in old.

$$\det \underline{\underline{T}}^f = \det \underline{\underline{S}} \det \underline{\underline{T}}^e \det \underline{\underline{S}}^{-1}$$
$$= \det \underline{\underline{T}}^e$$

Note $\det \underline{\underline{S}} = 1 / \det \underline{\underline{S}}^{-1}$

<u>Trace</u> of a matrix is sum of diagonal elements

$$\mathrm{Tr}\, T = \sum_i T_{ii}$$
$$\mathrm{Tr}(T_1 T_2) = \mathrm{Tr}(T_2 T_1)$$
$$\mathrm{Tr}\, T^f = \mathrm{Tr}\, S T^e S^{-1} = \mathrm{tr}\, T^e S S^{-1} = \mathrm{Tr}\, T^e$$

<u>Eigenvectors and Eigenvalues</u>

If $\hat{T} |\alpha\rangle = \lambda |\alpha\rangle$

$$\underline{\underline{T}}\, \underline{a} = \lambda\, \underline{a}$$

or $$\left(\underline{\underline{T}} - \lambda \underline{\underline{I}}\right) \underline{a} = \underline{0}$$

only possible if

$$\det \left(\underline{\underline{T}} - \lambda \underline{\underline{I}}\right) = 0$$

$$0 = \begin{vmatrix} T_{11} - \lambda & T_{12} & \cdots & T_{1n} \\ T_{21}, & T_{22} - \lambda, & T_{23}, & \cdots & T_{2n} \\ & & & \\ T_{n1} & & & T_{nn} - \lambda \end{vmatrix}$$

This is a polynomial of degree n $\quad P(\lambda) = 0$

$$c_n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots c_1 \lambda + c_0 = 0$$

<u>Hermitian transformation</u>

$$T^\dagger = \tilde{T}^*$$

$$\langle \hat{T}^\dagger \alpha | \beta \rangle = \langle \alpha | \hat{T} \beta \rangle$$

$$\langle \alpha | \hat{T} | \beta \rangle = \underline{a}^\dagger \underline{\underline{T}} \, \underline{b}$$

$$= (T^\dagger a)^\dagger b = \langle T^\dagger \alpha | \beta \rangle$$