2/14/06

Lecture 14 Formalism, Linear algebra

Read sections 3.1 and 3.2. Note much of 3.1 may be review

Linear algebra
Used for formalism of QM
Review for many. If section 3.1 is largely new let me know by email...

Vector space

consists of a set of vectors $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$ ... and a set of scalars a, b, c

can add two vectors

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

(by adding components)

or multiply a vector by a scalar

$$a|\alpha\rangle = |\gamma\rangle$$

A vector space is closed under vector addition and scalar multiplication

If $|\alpha\rangle$ and $|\beta\rangle$ are in space then so is $|\alpha\rangle + |\beta\rangle$ and $a|\alpha\rangle$ for any a

A linear combination of the vectors $|\alpha\rangle, |\beta\rangle, |\gamma\rangle$ is an expression of the form

$$a|\alpha\rangle + b|\beta\rangle + c|\gamma\rangle + \ldots$$

A vector $|\lambda\rangle$ is linearly independent of the set $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \ldots$ if it cannot be written as a linear combination of them

A collection of vectors is said to span the space if every vector can be written as a linear combination of the members in the set.

A set of linearly independent vectors that spans the space is called a basis. The number of vectors in any basis is called the dimension of the space.

With respect to a prescribed basis

$$|e_1\rangle, |e_2\rangle \ldots |e_n\rangle$$

any given vector

$$|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \ldots a_n|e_n\rangle$$

can be uniquely represented by the n-tuple of its components

$$|\alpha\rangle \longleftrightarrow (a_1, a_2, \ldots, a_n)$$

$$|\alpha\rangle + |\beta\rangle \longleftrightarrow a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$$

$$c|\alpha\rangle \longleftrightarrow (ca_1, ca_2, \ldots, ca_n)$$

null vector is all zeros

$$|0\rangle \longleftrightarrow (0, 0, \ldots, 0)$$

Easy to work with components. However they depend on choice of basis.

Inner Products (like dot product)

Inner product of two vectors is a complex number

$\langle \beta | \alpha \rangle = \langle \alpha | \beta \rangle^*$

$\left(= \sum_{i=1}^{n} b_i^* a_i \quad \text{for components} \right)$

$\langle \alpha | \alpha \rangle \geq 0$ and $\langle \alpha | \alpha \rangle = 0 \Rightarrow |\alpha\rangle = 0$

$\langle \alpha | \left(b|\beta\rangle + c|\gamma\rangle \right) = b\langle \alpha | \beta \rangle + c\langle \alpha | \gamma \rangle$

A vector space with an inner product is an <u>inner product space</u>

<u>norm</u> of a vector (length)

$\|\alpha\| = \sqrt{\langle \alpha | \alpha \rangle}$

Two vectors are <u>orthogonal</u> if

$\langle \alpha | \beta \rangle = 0$

<u>Orthonormal set</u> $\quad \langle \alpha_i | \alpha_j \rangle = \delta_{ij}$

Almost always convient to use a orthonormal basis

Components of a vector w.r.t a given basis are inner products

$|\alpha\rangle = \sum_i a_i |e_i\rangle$

$a_i = \langle e_i | \alpha \rangle$

For orthonormal basis

Schwarz inequality

Cos of angle between two vectors is less than or equal to one

$$|\langle \alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$$

Can define

$$\cos\theta \equiv \sqrt{\frac{\langle\alpha|\beta\rangle\ \langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle\ \langle\beta|\beta\rangle}}$$

note this could be for vectors which live in many dimension spaces

You will prove it in a problem. We will use it to prove H.U.P.

Linear transformations

$$\hat{T}|\alpha\rangle = |\gamma\rangle$$

Take one vector and gives you another vector. T is linear

$$\hat{T}(a|\alpha\rangle + b|\beta\rangle) = a\hat{T}|\alpha\rangle + b\hat{T}|\beta\rangle$$

Can think of $\hat{T}$ as a matrix. It takes a basis vector and gives you another vector in the space

$$\hat{T}|e_1\rangle = |\gamma_1\rangle$$

This vector $|\gamma_1\rangle$ has components

$$\hat{T}|e_1\rangle = |\gamma_1\rangle = T_{11}|e_1\rangle + T_{21}|e_2\rangle + \dots T_{n1}|e_n\rangle$$

$$\hat{T}|e_j\rangle = \sum_{i=1}^{n} T_{ij}|e_i\rangle \qquad j = 1, 2, \dots n$$

If $|\alpha\rangle = \sum_{i=1}^{n} a_i |e_i\rangle$

then $\hat{T}|\alpha\rangle = \sum_{j=1}^{n} \sum_{i=1}^{n} a_j \, T_{ij} |e_i\rangle$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij} \, a_j \, |e_i\rangle$$

$$\hat{T}|\alpha\rangle = |\alpha'\rangle$$

$|\alpha'\rangle$ has components $a'_i = \sum_{j=1}^{n} T_{ij} \, a_j$

Can work with matricies

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$ $n \times 1$ column matrix

$$\underline{\underline{T}} = n \times n \text{ square matrix} \quad \begin{pmatrix} T_{11} & T_{21} & \cdots & T_{n1} \\ T_{12} & & & T_{n2} \\ \vdots & & & \vdots \\ T_{1n} & & \cdots & T_{nn} \end{pmatrix}$$

<u>Transpose</u> of a column matrix is a <u>row matrix</u>

$$\tilde{a} = (a_1, a_2, \ldots, a_n)$$

A square matrix is symmetric if

it is equal to its transpose $T_{ij} = T_{ji}$

Symmetric $\tilde{T} = T$

Antisymmetric $\tilde{T} = -T$ $T_{ij} = -T_{ji}$

<u>Conjugate</u> of a matrix is complex conjugate of each element

$$\underline{\underline{T}}^* = \begin{pmatrix} T_{11}^* & T_{12}^* & \cdots & T_{1n}^* \\ T_{n1}^* & & & T_{nn}^* \end{pmatrix}$$

$$\underline{a}^* = \begin{pmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{pmatrix}$$

Real $\underline{\underline{T}}^* = \underline{\underline{T}}$ Imaginary $\underline{\underline{T}}^* = -\underline{\underline{T}}$

<u>Hermitian conjugate</u> or <u>adjoint</u> of a matrix is the transposed conjugate

$$\underline{\underline{T}}^\dagger = \tilde{T}^* = \begin{pmatrix} T_{11}^* & T_{21}^* & T_{n1}^* \\ T_{1n}^* & & T_{nn}^* \end{pmatrix}$$

$$a^\dagger = \tilde{a}^* = (a_1^*, a_2^*, \ldots a_n^*)$$

inner product

$$\langle \alpha | \beta \rangle = \underline{a}^\dagger b = (a_1^*, a_2^*, \ldots, a_n^*) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$