2/9/01

Lec. 14 Moving Particle in a Box

Review particle in a box

$$V = \begin{cases} 0 & 0 < x < a \\ \infty & \text{else} \end{cases}$$

$$H \psi_n = E_n \psi_n \qquad H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \quad \text{inside}$$

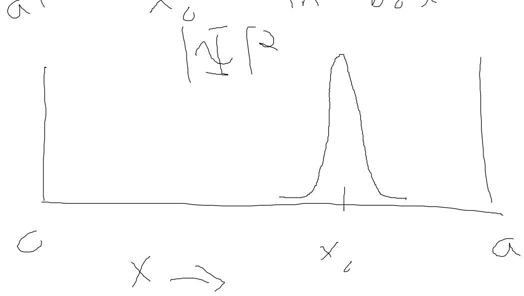
$$\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2 = \hbar\omega n^2 \qquad \omega \equiv \frac{\hbar \pi^2}{2ma^2}$$

Inital condition $\Psi(x,0)$ given.

Example particle localized at x_0 in box

$$\Psi = N e^{-q^2 (x - x_0)^2}$$



assume $e^{-q^2 x_0^2} \ll 1$
and $e^{-q^2 (a-x_0)^2} \ll 1$

So q can't be to small otherwise wave function will leak out box.

$$1 = \int_0^a dx\, |\Psi(x,0)|^2 = N^2 \int_0^a dx\, e^{-2q^2(x-x_0)^2}$$

$$\simeq N^2 \int_{-\infty}^{\infty} dx\, e^{-2q^2(x-x_0)^2} = N^2 \sqrt{\frac{\pi}{2q^2}}$$

$$N = \left(\frac{2q^2}{\pi}\right)^{1/4}$$

$$\Psi(x,0) = \left(\frac{2q^2}{\pi}\right)^{1/4} e^{-q^2(x-x_0)^2}$$

Expand Ψ in eigenstates

$$\Psi(x,0) = \sum_i c_i \psi_i(x)$$

Use orthogonality to find c_i

$$c_i = \int_0^a \psi_i^*(x)\, \Psi(x,0)$$

$$c_i = \int_{-\infty}^{\infty} dx \sqrt{\frac{2}{a}} \sin\left(\frac{i\pi x}{a}\right) e^{-q^2(x-x_0)^2} \left(\frac{2q^2}{\pi}\right)^{1/4}$$

$$= \left(\frac{8q^2}{\pi a^2}\right)^{1/4} \int_{-\infty}^{\infty} dx\, e^{-q^2(x-x_0)^2} \sin\left[\frac{i\pi}{a} x\right]$$

let $t = x - x_0$ $\quad x = t + x_0$

$$c_i = \left(\frac{8q^2}{\pi a^2}\right)^{1/4} \int_{-\infty}^{\infty} dt\, e^{-q^2t^2} \sin\left[\frac{i\pi}{a}(t + x_0)\right]$$

$$\int_{-\infty}^{\infty} e^{-q^2t^2} \sin[p(t+\lambda)]\, dt = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \sin p\lambda$$

From integral tables

$$c_i = \left(\frac{8q^2}{\pi a^2}\right)^{1/4} \frac{\sqrt{\pi}}{q} e^{-\left(\frac{i\pi}{a}\right)^2 \frac{1}{4q^2}} \sin\left(\frac{i\pi}{a} x_0\right)$$

$$\boxed{c_i = \left(\frac{8\pi}{q^2 a^2}\right)^{1/4} e^{-\frac{\pi^2 i^2}{4q^2}} \sin\left(\frac{i\pi x_0}{a}\right)}$$

Check normalization $\sum_i |c_i|^2 = 1$

Time dep.

$$\Psi(x,t) = \sum_j c_j \psi_j(x)\, e^{-iE_j t/\hbar}$$

$$= \left(\frac{8\pi}{q^2a^2}\right)^{1/4} \left(\frac{2}{a}\right)^{1/2} \sum_j e^{-\frac{\pi^2 j^2}{4q^2}} \sin\left(\frac{j\pi x_0}{a}\right) \sin\left(\frac{j\pi x}{a}\right) e^{-i j^2 \omega t}$$

Let $a = 1$. measure length in units of a

$q = 10$ measure time in units of $1/\omega$

$x_0 = .5$ $\omega = \frac{\hbar \pi^2}{2ma^2}$