2/7/01

Lecture 13 Review So far

Last time: A narrow finite square well becomes a delta function in the limit

$$V = \begin{cases} -V_0, & -a < x < a \\ 0 \end{cases}$$

$$\lim a \to 0,\ V_0 \to \infty,\ V_0 \cdot 2a = \alpha \quad \text{finite}$$

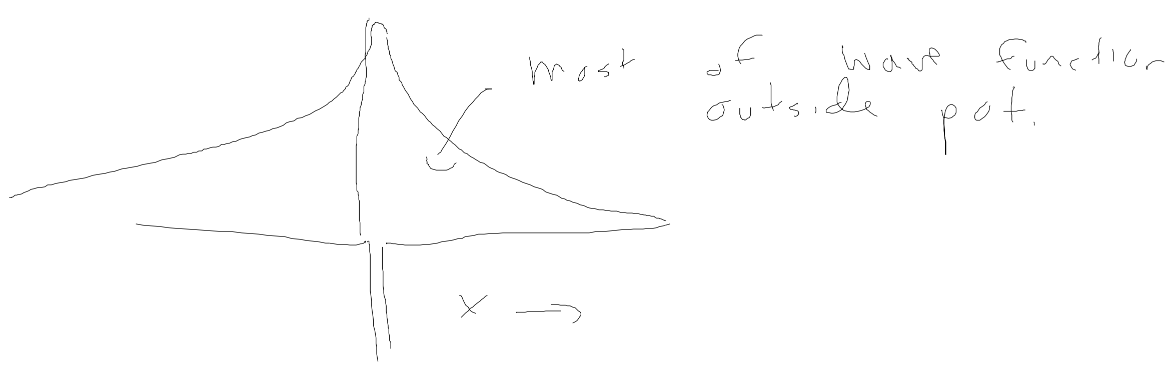
$$V \to -\alpha\,\delta(x)$$

and we showed the energy of the bound state is

$$E = -\frac{m\,2V_0^2 a^2}{\hbar^2} = \frac{-m\alpha^2}{2\hbar^2}$$

There is only one bound state. Another bound state would need to have one more node. However there is no room for the node as the square well becomes very narrow

Note for a narrow square well or for a delta function the particle spends all of its time outside potential in a force free region with $V \equiv 0$.



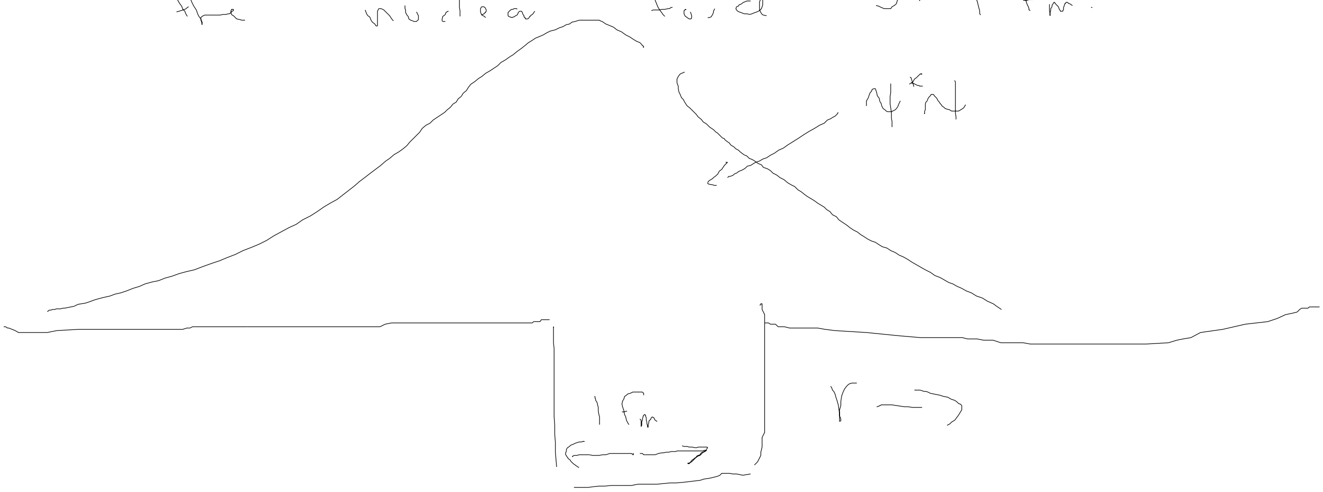
Classically if particle ever got into a region with $V = 0$ it would escape to ∞

Example: Deuteron is a weakly bound state of a neutron and a proton

$$\langle r^2 \rangle^{1/2} \approx 2.2 \text{ Fm}$$

$$1 \text{ Fm} = 10^{-15} \text{ meters}$$

This is greater than the range of the nuclear force ~ 1 Fm.



Review so far

Energy eigenstates $\hat{H} \psi_n = E_n \psi_n$

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \quad \text{Hamiltonian}$$

$\psi_n(x)$ = Energy eigenstates or stationary states time dep. $e^{-iE_n t/\hbar}$

Eigenstates are orthonormal $\int_{-\infty}^{\infty} dx\, \psi_n^*(x) \psi_m(x) = \delta_{nm}$

Eigenstates are complete. Can expand any function

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

Note if there are also scattering states need to integrate rather than sum

$$\Psi(x,0) = \int_{-\infty}^{\infty} \phi(k)\, \psi_k(x)\, dk$$

Example free particle $c_n \to \phi_k$

$$\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}$$

Finally many pot. have both scattering and bound states

$$\Psi(x,0) = \underbrace{\sum_n c_n \psi_n(x)}_{\text{bound}} + \underbrace{\int dn\, c(n)\, \psi_n(x)}_{\text{scattering}}$$

Time dep. of state with energy E_n is $e^{-iE_nt/\hbar}$

$$\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_nt/\hbar}$$

Normalization

$$\int \psi_n^*(x)\psi_n(x)\, dx = 1 \qquad \text{eigenstates}$$

$$\int \Psi^*(x,t)\, \Psi(x,t)\, dx = 1 \qquad \text{full wave function for all time}$$

$$\Rightarrow \sum_n c_n^* c_n = 1$$

$c_n^* c_n = P_n$ = prob. to be in n^{th} energy eigenstate

$$\int dk\ \phi^*(k)\,\phi(k) = 1$$

$\Psi(x,t)$ is coordinate space wave func.

$\Psi^*\Psi\,dx$ is prob. to find particle between x and $x+dx$

$\phi(k)$ is momentum space wave function

$\phi^*(k)\,\phi(k)\,dk$ is prob. to find particle with momentum between $\hbar k$ and $\hbar(k+dk)$

<u>Different solutions</u>

H. Osc. $V = \frac{1}{2}kx^2$: $\psi_n = H_n(\xi)\,e^{-\xi^2/2}$ $\quad E_n = (n+\frac{1}{2})\hbar\omega$

Infinite square well: $\psi_n = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$ $\quad E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$

Free particle: $\psi(x) = \frac{1}{\sqrt{2\pi}}\int dk\ \phi(k)\,e^{ikx}$

δ pot. $V = -\alpha\,\delta(x)$ $\quad \psi = e^{-\kappa|x|}$, $\quad E = -\frac{\hbar^2\kappa^2}{2m}$

$$E = -\frac{m\alpha^2}{2\hbar^2}$$

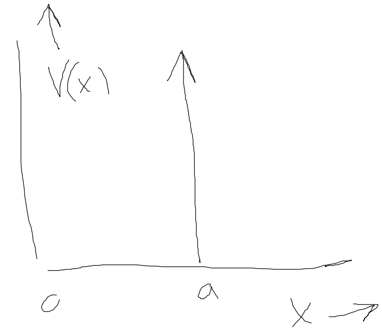
Start to read Chap. 3 FORMALISM

<u>Example</u> Problem 2.46

$V = \infty \qquad x < 0$

$V = \alpha\,\delta(x-a) \qquad x \geq 0$

This is a box but particle can leak out of box to the right



Have a particle start in the well and gradually leak out

$0 < x < a$ $\quad \psi = Ae^{ikx} + Be^{-ikx}$

$V = 0$ $\quad E = \frac{\hbar^2 k^2}{2m}$

$a < x < \infty$ $\quad \psi = Ce^{ikx} + De^{-ikx}$

$E = \frac{\hbar^2 k^2}{2m}$ still $V = 0$

But De^{-ikx} has momentum $-\hbar k$ and represents a particle going to the left and a comming in from $x = +\infty$

So only keep Ce^{ikx} which represents a particle leaving the well.

Example $\quad {}^{238}U \rightarrow {}^{234}Th + {}^{4}He$

Radio active decay of uranium. Note this takes $t_{1/2} = 4.5 \times 10^9$ years

B.C. $\quad \psi(0) = 0$

$\psi(a)$ is cont.

$\psi'(a)$ is discont.

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + \alpha\,\delta(x)\psi = E\psi$$

integrate from $a-\epsilon$ to $a+\epsilon$

$$-\frac{\hbar^2}{2m}\left[\frac{\partial \psi}{\partial x}(a+\epsilon) - \frac{\partial \psi}{\partial x}(a-\epsilon)\right] + \alpha\,\psi(a) = 0$$

$$\boxed{\frac{d\psi}{dx}(a+\epsilon) - \frac{d\psi}{dx}(a-\epsilon) = \left(\frac{2m\alpha}{\hbar^2}\right)\psi(a)}$$

So $B = -A$ for $\psi(0) = 0$

$$\psi(a-\epsilon) = A\left(e^{ika} - e^{-ika}\right)$$

$$\psi(a+\epsilon) = C e^{ika}$$

$$A\left(e^{ika} - e^{-ika}\right) = C e^{ika}$$

$$-ik\,A\left(e^{ika} + e^{-ika}\right) + ik\,C e^{ika} = \frac{2m\alpha}{\hbar^2} C e^{ika}$$

$$ik\left(A e^{ika} - e^{-ika}\right) - ik\left(e^{ika} + e^{-ika}\right)A = \frac{2m\alpha}{\hbar^2} A\left(e^{ika} - e^{-ika}\right)$$

$$\cancel{ik A e^{2ika}} - \cancel{ik e^{2ika}} A - 2ikA = \frac{2m\alpha}{\hbar^2} A\left(e^{2ika} - 1\right)$$

$$1 = \frac{m\alpha}{\hbar^2 ik}\left(1 - e^{2ika}\right)$$

$$\boxed{1 = \left(\frac{mi\alpha}{\hbar^2 k}\right)\left(e^{2ika} - 1\right)}$$

Implicit equation for k solution is complex