Lecture 13 Review So Far

Last time: A narrow finite square well becomes a delta function in the limit
\[ V = \begin{cases} V_0, & -\alpha < x < \alpha \\ 0 & \end{cases} \]

In \( \alpha \to 0 \), \( V_0 \to \infty \), \( V_0 \cdot 2\alpha = \infty \) finite
\( V \to -\alpha \delta(x) \)

And we showed the energy of the bound state is
\[ E = -\frac{m}{2} \frac{2V_0 \alpha^2}{\hbar^2} = \frac{-m \alpha^2}{2 \hbar^2} \]

There is only one bound state. Another bound state would need to have one more node. However, there is no room so, the node as the square well becomes very narrow.

Note for a narrow square well or for a delta function the particle spends all of its time outside the region potential in a force-free region most of wave function outside pot.
Classically if a particle were to get into a region with \( V = 0 \) it would escape to \( \infty \).

Example: Deuteron is a weakly bound state of a proton and a neutron:
\[
\langle r^2 \rangle^{1/2} \approx 2.2 \text{ Fm}
\]

1 Fm = 10^{-15} meters

This is greater than the range of the nuclear force \( \approx 1 \text{ Fm} \).

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Review so far:

Energy eigenstates:
\[
\hat{H} \Psi_n = E_n \Psi_n
\]

\[
\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V(x) \quad \text{Hamiltonian}
\]

\[\Psi_n(x)\] - Energy eigenstates or stationary states \( \propto \text{time dep. } e^{\pm iEt/\hbar} \)

Eigenstates are orthonormal:
\[
\int \Psi_n^*(x) \Psi_m(x) \, dx = \delta_{nm}
\]
Eigenstates are complete. Can expand any function

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

Node if these are also scattering states need to integrate rather than sum

$$\Psi(x,0) = \int_{-\infty}^{\infty} \phi(k) \hat{\Psi}(x) \, dk$$

Example free particle $c_n \rightarrow \frac{1}{k}$

$$\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}$$

Finally, many not have both scattering and bound states

$$\Psi(x,0) = \sum_{n} c_n \psi_n(x) + \sum_{k} c_k \psi_k(x)$$

bound  scattering

Time dep. of state with energy $E_n$ is $e^{-iE_nt/\hbar}$

$$\Psi(x,t) = \sum_{n} c_n \psi_n(x) e^{-iE_nt/\hbar}$$

Normalization

$$\int \psi_n^*(x) \psi_n(x) \, dx = 1 \quad \text{eigenstates}$$

$$\int \Psi^*(x,t) \Psi(x,t) \, dx = 1 \quad \text{full wave function}$$

$$\sum_{n} c_n^* c_n = 1 \quad \text{for all time}$$

$$c_n^* c_n = P_n = \text{Prob. to be in n'th energy eigenstate}$$
\[ \int_{-\infty}^{\infty} \psi^* \psi \, dx = 1 \]

\( \psi(x,t) \) is coordinate space wave function.

\( \int_{x}^{x+dx} \psi^* \psi \, dx \) is prob. to find particle between \( x \) and \( x + dx \).

\( \phi(k) \) is momentum space wave function

\( \phi^* \phi \, dk \) is prob. to find particle with momentum between \( k \) and \( k + dk \).

Different solutions

H. Osc.: \( V = \frac{1}{2} k x^2 \):
\[ \psi_n = H_n(kx) e^{-\frac{k^2 x^2}{2}} \]
\[ E_n = (n + \frac{1}{2}) \hbar \omega \]

Infinite Square Well:
\[ \psi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \]
\[ E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \]

Free particle:
\[ \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} \, dk \]

S. pot.
\[ V = -\alpha S(x) \]
\[ \psi = e^{-\frac{1}{2}c|x|} \]
\[ E = -\frac{m\alpha^2}{2\hbar^2} \]

Start to read Chp. 3 

Example Problem 2.46
\[ V = \infty \quad x < 0 \]
\[ V = \alpha S(x-a) \quad x \geq 0 \]

This is a box but particle can leak out of box to the right.
Have a particle start in the well and gradually leak out

\[ 0 < x < a \] \quad \Psi(x) = A e^{ikx} + Be^{-ikx} \quad V = 0 \quad E = \frac{p^2}{2m} \]

\[ a < x < \infty \] \quad \Psi(x) = C e^{ikx} + Ne^{-ikx} \quad E = \frac{p^2}{2m} \quad \text{still} \quad V = 0 \]

But \( e^{-ikx} \) has momentum \( -tk \) and represents a particle going to the left and a comoving incoming from \( x = \infty \)

So only keep \( Ce^{ikx} \) which represents a particle leaving the well.

Example: \( ^{238}\text{U} \rightarrow ^{234}\text{Th} + ^{4}\text{He} \)

Radiactive decay of uranium. Note this takes \( t_{1/2} = 4.5 \times 10^9 \) years

B.C. \( \Psi(0) = 0 \)

\[ \Psi(a) \text{ is cont.} \]

\[ \Psi'(a) \text{ is discont.} \]

\[ -\frac{t^2}{2m} \frac{d^2\Psi}{dx^2} + \alpha s(x)\Psi = E\Psi \]

Integrate from \( \alpha - \epsilon \) to \( \alpha + \epsilon \)

\[ -\frac{t^2}{2m} \left[ \frac{d\Psi}{dx}(\alpha + \epsilon) - \frac{d\Psi}{dx}(\alpha - \epsilon) \right] + \alpha \Psi(a) = 0 \]
\[
\frac{\partial^2 \psi}{\partial x^2} (a+c) - \frac{\partial^2 \psi}{\partial x^2} (a-c) = \left( \frac{2m\alpha}{\hbar^2} \right) \psi (a)
\]

So \( B = -A \quad \forall \alpha \) \( \psi (a) = 0 \)

\[\psi (a-c) = A (e^{ika} - e^{-ika}) \]
\[\psi (a+c) = Ce^{ika} \]

\[A (e^{ika} - e^{-ika}) = Ce^{ika} \]
\[-iK A (e^{ika} + e^{-ika}) + iK Ce^{ika} = 2m\alpha \frac{Ce^{ika}}{\hbar^2} \]
\[iK (A (e^{ika} - e^{-ika}) - iK (e^{ika} + e^{-ika}) A = 2m\alpha \frac{A(e^{ika} - e^{-ika})}{\hbar^2} \]
\[iK A e^{ika} - iK e^{-ika} A - 2iK A = 2m\alpha \frac{A e^{ika}}{\hbar^2} \]

\[I = \frac{m\alpha}{\hbar^2} \frac{1 - e^{2ika}}{1 - e^{2ika}} \]

\[1 = (\frac{m\alpha}{\hbar^2} iK)(e^{2ika} - 1) \]

Implicit equation for \( \epsilon \) solution is complex