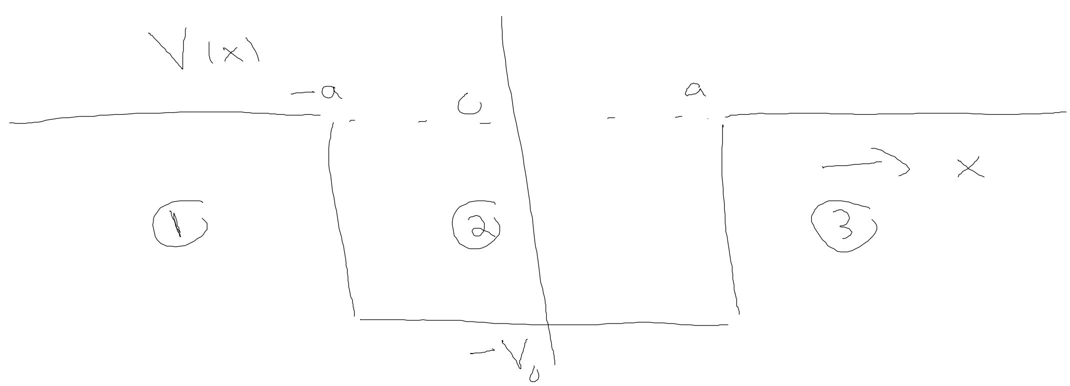
2/5/01

Lecture 12 Finite Square Well

Read discussion of scattering states for a delta func. pot.

Read Sec 2.6

$$V(x) = \begin{cases} -V_0 & -a < x < a \\ 0 & \text{else} \end{cases}$$



Look for bound state $E < 0$

In region ① $V = 0$ $\psi = A e^{+\kappa x}$ so $\psi \to 0$ as $x \to -\infty$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = E\psi \qquad \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

Just like for delta function

In region ③ $V = 0$ $\psi = B e^{-\kappa x}$
$\psi \to 0$ as $x \to +\infty$

In region ② $V = -V_0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} - V_0\psi = E\psi$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\left[\frac{\sqrt{2m(V_0+E)}}{\hbar}\right]^2 \psi$$

Assume $V_0 + E > 0$ even though $E < 0$

Let $l = \frac{\sqrt{2m(E+V_0)}}{\hbar}$

$\psi = C\sin(lx) + D\cos lx$

Given a solution to S. eq. $\psi(x)$
Then $\psi(-x)$ is also a solution

(1) $$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x) = E\psi(x)$$

let $z = -x$ $\quad \partial z = -\partial x \quad \partial^2 z = \partial x^2$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi(-z) + V(-z)\psi(-z) = E\psi(-z)$$

Pot is symmetric $V(-z) = V(z)$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi(-z) + V(z)\psi(-z) = E\psi(-z)$$

let $z \to x$ (2) $$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(-x)}{\partial x^2} + V(x)\psi(-x) = E\psi(-x)$$

so if $\psi(x)$ is a solution then $\psi(-x)$ is also a solution with same energy

add (1) + (2)

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\psi(x) + \psi(-x)\right] + V(x)\left[\psi(x) + \psi(-x)\right] = E\left[\psi(x) + \psi(-x)\right]$$

Now $\psi(x) + \psi(-x)$ is an even wave function.

Likewise subtract ① and ②

$\Rightarrow$ Any solution to S. eq. for a pot. which is an even function can be taken to be even or odd.

Consider even solutions for square well.

$$\psi = \begin{cases} A e^{-\kappa |x|} & |x| > a \\ D \cos l x & |x| < a \end{cases}$$

B.C. (1) ~~derivative~~ wave function is cont.

(2) $\partial\psi/\partial x$ is cont. at $x = a$

① $A e^{-\kappa a} = D \cos l a$

② $-\kappa A e^{-\kappa a} = -l D \sin l a$

Divide ② by ①

$$\boxed{\kappa = l \tan l a}$$

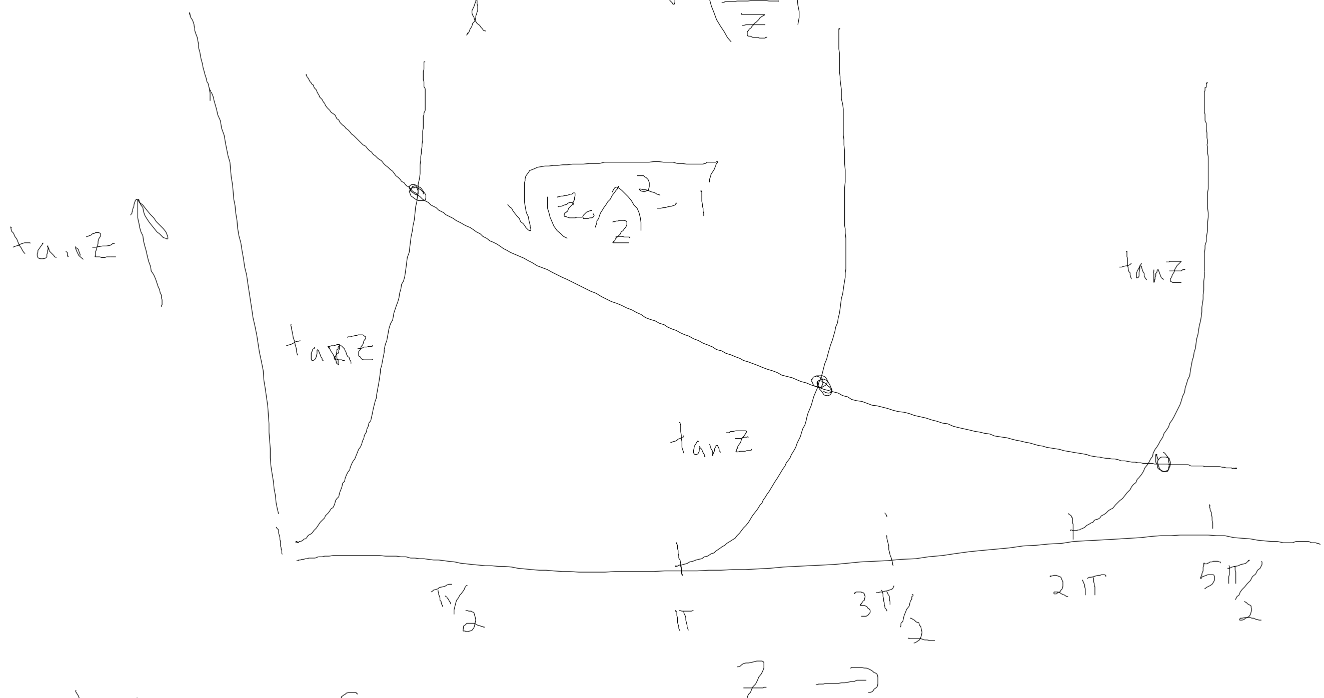
$$\kappa = \frac{\sqrt{-2mE}}{\hbar} \qquad l = \frac{\sqrt{2m(V_0 + E)}}{\hbar}$$

m, a, V_0 are known only unknown is E. Trans. eq. for E

let $Z = la$

$Z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$ $\qquad \kappa = \frac{\sqrt{-2mE}}{\hbar}$

$$\tan Z = \frac{\kappa}{l} = \sqrt{\left(\frac{Z_0}{Z}\right)^2 - 1}$$



let $Z_0 = 8$

Note $\quad l = Z/a \qquad Z^2/a^2 = l^2 = \frac{2m(V_0+E)}{\hbar^2}$

$Z_0^2 = \frac{a^2}{\hbar^2}2mV_0$

$$\kappa^2 = -\frac{2mE}{\hbar^2}$$

$$Z_0^2 - Z^2 = a^2\left[\frac{2mV_0}{\hbar^2} - \frac{2m(V_0+E)}{\hbar^2}\right] = \frac{a^2(-2mE)}{\hbar^2}$$

so $$\left(\frac{Z_0^2}{Z^2} - 1\right)^{1/2} = \frac{(Z_0^2 - Z^2)^{1/2}}{Z} = \frac{a}{Z}\frac{(-2mE)^{1/2}}{\hbar}$$

$$= \frac{\kappa}{l}$$

so $$\left[\left(\frac{Z_0}{Z}\right)^2 - 1\right]^{1/2} = \tan Z$$

Deep well. Consider limit $z_0 \to \infty$

$$\tan z \to \infty$$

$$z = n \frac{\pi}{2} \qquad \text{with } n \text{ odd}$$

$$z^2 = l^2 a^2 = \frac{2m(V_0 + E)}{\hbar^2} a^2$$

$$E = \frac{\hbar^2 z^2}{2 m a^2} \quad V_0$$

$$E_n = -V_0 + \frac{n^2}{4} \frac{\hbar^2 \pi^2}{2 m a^2} \qquad n \text{ odd}$$

Infinite square well result for $a_{tot} = 2a$

Shallow or narrow well

As $V_0 \to 0$ $\qquad E \to 0$ from below

$$l = \frac{\sqrt{2m(V_0 + E)}}{\hbar} \simeq \frac{\sqrt{2mV_0}}{\hbar}$$

$$\kappa = \frac{\sqrt{-2mE}}{\hbar} = l \tan la \approx \frac{\sqrt{2mV_0}}{\hbar} a$$

$$\kappa \simeq l^2 a = \frac{2mV_0}{\hbar^2} a$$

$$\frac{\sqrt{-2mE}}{\hbar} = \frac{2mV_0 \, a}{\hbar^2}$$

$$-2mE = \frac{4mV_0^2 \, a^2}{\hbar^2}$$

$$\boxed{E = -\frac{2mV_0^2 a^2}{\hbar^2}} = -\frac{m\alpha^2}{2\hbar^2}$$

Note $\int_{-\infty}^{\infty} V(x)\, dx = V_0 \, 2a = \alpha$