2/2/01

Lecture 11 Delta Function Cont.

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

and

$$\int_a^b dx\, \delta(x) = 1 \quad \text{if} \quad a < 0 < b$$

<u>Problem 2.23</u> Evaluate the following

(a) $\int_{-3}^{1} (x^3 - 3x^2 + 2x - 1)\, dx\, \delta(x+2) = (x^3 - 3x^2 + 2x - 1)\Big|_{-2}$

$= -8 - 3 \cdot 4 - 4 - 1 = -25$

(b) $\int_0^\infty [\cos 3x + 2]\, \delta(x - \pi)\, dx = \cos(3\pi) + 2 = 1$

(c) $\int_{-1}^{1} \exp[\,|x| + 3\,]\, \delta(x-2)\, dx = 0$

<u>Delta Func. Pot. cont.</u>

$$V = -\alpha\, \delta(x)$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} - \alpha\, \delta(x)\, \psi = E \psi$$

If $x \neq 0$ $\quad \frac{\partial^2 \psi}{\partial x^2} = -\frac{2mE}{\hbar^2} \psi = \kappa^2 \psi$

$$\kappa = \frac{\sqrt{-2mE}}{\hbar} \qquad E < 0 \text{ for bound state}$$

$$\boxed{E = -\frac{\hbar^2 \kappa^2}{2m}}$$

$$\psi = A e^{-\kappa x} + B e^{\kappa x}$$

for $x < 0$ $\quad \psi = B e^{\kappa x}$

for $x > 0$ $\quad \psi = A e^{-\kappa x}$ $\quad$ so $\psi \to 0$ as $|x| \to \infty$

Boundary conditions at $x=0$

1) ψ is cont. $\Rightarrow \psi = A e^{-\kappa|x|}$

2) $\frac{\partial \psi}{\partial x}$ is cont. except where $V=\infty$

But $V=\infty$ at 0 so $\frac{\partial \psi}{\partial x}$ not cont.

To find discont. in $\frac{\partial \psi}{\partial x}$ integrate S eq.

Integrate from $-\epsilon$ to $+\epsilon$ and take limit $\epsilon \to 0$

$$\int_{-\epsilon}^{\epsilon} dx \left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}\right) + \int_{-\epsilon}^{\epsilon} dx\,(-\alpha)\,\delta(x)\,\psi(x)$$

$$= E\int_{-\epsilon}^{\epsilon} dx\,\psi(x)$$

$$-\frac{\hbar^2}{2m}\left[\frac{\partial \psi}{\partial x}(+\epsilon) - \frac{\partial \psi}{\partial x}(-\epsilon)\right] - \alpha\,\psi(0)$$

$$= 2\epsilon E\,\psi(0) \to 0 \text{ as } \epsilon \to 0$$

$$\boxed{\frac{\partial \psi}{\partial x}(+\epsilon) - \frac{\partial \psi}{\partial x}(-\epsilon) \equiv \Delta\psi' = -\frac{2m\alpha}{\hbar^2}\psi(0)}$$

Discont. in ψ' is related to $\psi(0)$ and strength α of delta function.

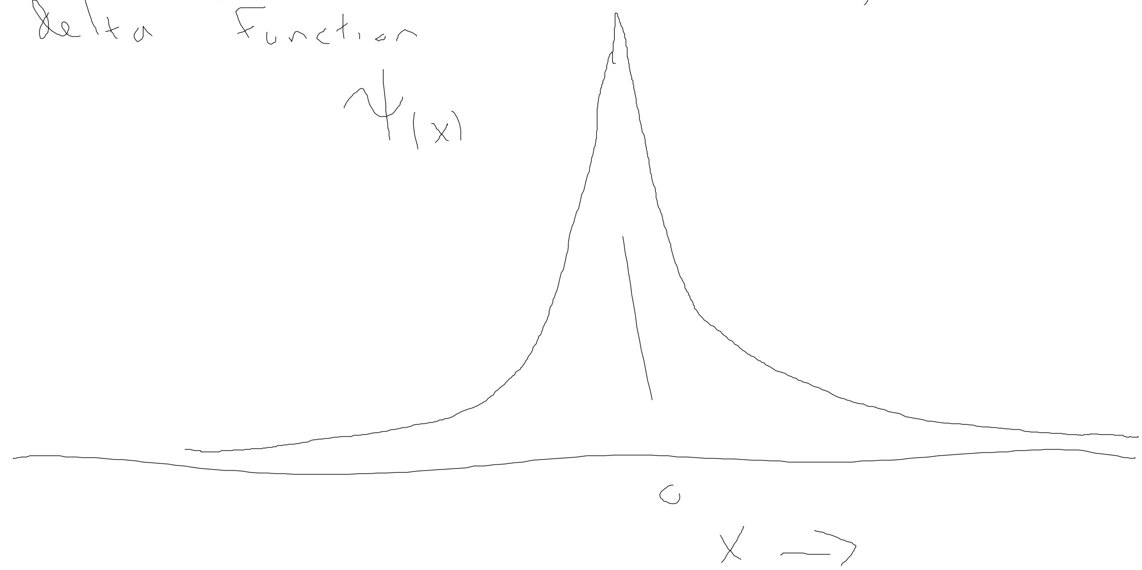
$\psi = A e^{-\kappa|x|}$ $\qquad \psi(0) = A$

$\psi'(+\epsilon) = -\kappa A$ $\qquad \psi'(-\epsilon) = +\kappa A$

$$\Delta\psi' = -2\kappa A = -\frac{2m\alpha}{\hbar^2}A$$

$$\kappa = \frac{m\alpha}{\hbar^2} \qquad \boxed{E = -\frac{m\alpha^2}{2\hbar^2} = -\frac{\hbar^2\kappa^2}{2m}}$$

There is one bound state. The cusp in the wave function at the origin is related to the strength of the delta function



Normalize

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = |A|^2 \int_{-\infty}^{\infty} e^{-2\kappa|x|} \, dx$$

$$= 2|A|^2 \int_{0}^{\infty} e^{-2\kappa x} \, dx = \frac{2|A|^2}{2\kappa} = 1$$

$$A = \sqrt{\kappa} = \frac{\sqrt{m\alpha}}{\hbar}$$

$$\boxed{\psi = \frac{\sqrt{m\alpha}}{\hbar} e^{-\frac{m\alpha}{\hbar^2}|x|}} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$$

Read text about scattering states.

$\boxed{\text{Problem 2.25}}$ What is Fourier transform of a delta function?

We will use Plancherel's Theorem

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; F(k) \, e^{ikx}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; f(x) \, e^{-ikx}$$

Let $f(x) = \delta(x)$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) \, e^{-ikx} \, dx = \frac{1}{\sqrt{2\pi}}$$

Fourier transform of a delta function is a constant.

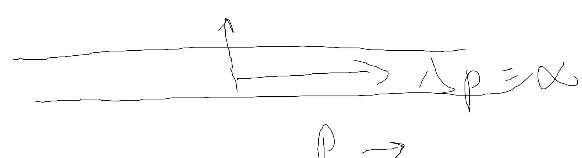
Note HUP

If $\psi^*\psi(x) \propto \delta(x)$

Uncert. in position $\Delta x = 0$

Therefore $\Delta p = \infty$

$$\phi^*\phi(k) \propto \frac{1}{\sqrt{2\pi}}$$



Fourier transforms build in HUP.

$$\boxed{F(k) = \frac{1}{\sqrt{2\pi}}}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; \frac{1}{\sqrt{2\pi}} \, e^{ikx}$$

$$\boxed{\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{ikx}}$$

This formula gives mathematicians apoplexy
Clearly infinite for $x = 0$
But integral does not converge for $x \neq 0$

These problems come because $\delta(x)$ does not satisfy conditions for Plancherel's theorem.

Need $f(x)$ to be normalizable

$$\int_{-\infty}^{\infty} dx \; |f(x)|^2 < \infty$$

Note $$\int_{-\infty}^{\infty} dx \; \delta(x)^2 \overset{??}{=} \delta(0) \overset{?}{=} \infty$$

Ugly mess

However if care is taken to define integral

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{ikx}$$

Can be very useful.